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А.Н.Васильев, Н.В.Антонов

ANOMALOUS SCALING IN TURBULENCE I.
Field-theoretical approach

Учебно-методическое пособие

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- Рецензент: профессор кафедры статистической физики физического факультета СПбГУ, д.ф.-м.н. Л.Ц.Аджемян.

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В учебно-методическом пособии с помощью методов квантовой теории поля рассматриваются процессы турбулентного переноса скалярного поля (примеси или температуры).
Пособие предназначено для студентов 4-7-го курсов, аспирантов, соискателей и других обучающихся по специальности теоретическая физика.

Abstract. The field theoretic renormalization group (RG) is applied to the problem of a passive scalar advected by the Gaussian self-similar velocity field with finite correlation time and in the presence of an imposed linear mean gradient. The energy spectrum in the inertial range has the form $E(k) \propto k^{1-\varepsilon}$, and the correlation time at the wavenumber k scales as $k^{-2+\eta}$. It is shown that, depending on the values of the exponents ε and η, the model in the inertial-convective range exhibits various types of scaling regimes associated with the infrared stable fixed points of the RG equations: diffusive-type regimes for which the advection can be treated within ordinary perturbation theory, and three nontrivial convection-type regimes for which the correlation functions exhibit anomalous scaling behavior. Explicit asymptotic expressions for the structure functions and other correlation functions are obtained; they are represented by superpositions of power laws with nonuniversal amplitudes and universal (independent of the anisotropy) anomalous exponents, calculated to the first order in ε and η in any space dimension. These anomalous exponents are determined by the critical dimensions of tensor composite operators built of the scalar gradients, and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behaviour. The leading terms of the even (odd) structure functions are given by the scalar (vector) operators. For the first nontrivial regime the anomalous exponents are the same as in the rapid-change version of the model; for the second they are the same as in the model with time-independent (frozen) velocity field. In these regimes, the anomalous exponents are universal in the sense that they depend only on the exponents entering into the velocity correlator. For the last regime the exponents are nonuniversal (they can depend also on the amplitudes); however, the nonuniversality can reveal itself only in the second order of the RG expansion. A brief discussion of the passive advection in the non-Gaussian velocity field governed by the nonlinear stochastic Navier-Stokes equation is also given.

1. Introduction

The investigation of intermittency and anomalous scaling in fully developed turbulence remains one of the major theoretical problems. Both the natural and numerical experiments suggest that the deviation from the predictions of the classical Kolmogorov–Obukhov theory is even more strongly pronounced for a passively advected scalar field than for the velocity field itself; see, e.g., [1, 2, 3, 4, 5, 6] and literature cited therein. At the same time, the problem of passive advection appears to be easier tractable theoretically: even simplified models describing the advection by a "synthetic" velocity field with prescribed Gaussian statistics reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments, see [3]–[38]. Therefore, the problem of a passive scalar advection, being of practical importance in itself, may also be viewed as a starting point in studying anomalous scaling in the turbulence on the whole.

Recently, a great deal of attention has been drawn by a simple model of the passive scalar advection by a self-similar Gaussian white-in-time velocity field, the so-called "rapid-change model," introduced by Kraichnan [10]; see [8–30] and references therein. For the first time, the anomalous exponents have been calculated on the basis of a

microscopic model and within regular expansions in formal small parameters. Within the "zero-mode approach" to the rapid-change model, developed in [14, 15, 16, 17], nontrivial anomalous exponents are related to the zero modes (homogeneous solutions) of the closed exact equations satisfied by the equal-time correlations. In this sense, the model is "exactly solvable." The anomalous exponents are universal, i.e., they depend only on the space dimension and the exponent entering into the velocity correlator.

Of course, the Gaussian character, isotropy, and time decorrelation are strong departures from the statistical properties of genuine turbulence. One step toward the construction of a more realistic model of passive advection is the account of the finite correlation time of the velocity field.

In [34, 35], a generalized phenomenological model was considered in which the temporal correlation of the advecting field was set by eddy turnover (see also an earlier work [36], where the probability distribution function in an analogous model was studied). It was argued that the anomalous exponents may depend on more details of the velocity statistics, than only the exponents. This idea has received some analytical support in [37], where the case of short but finite correlation time was considered for the special case of a local turnover exponent. The anomalous exponents were calculated within the perturbation theory with respect to the small correlation time, with Kraichnan's rapid-change model taken as the zeroth order approximation. The exponents obtained in [37] appear to be nonuniversal, through the dependence on the correlation time. The exact inequalities obtained in [38] using the so-called refined similarity relations also point up some significant differences between the zero and finite correlation-time problems.

In the paper [32], the field theoretic renormalization group (RG) and operator product expansion (OPE) were applied to the model [10]. The feature specific to the theory of turbulence is the existence in the corresponding field theoretical models of the composite operators with *negative* scaling ("critical") dimensions. Such operators are termed "dangerous," because their contributions to the OPE for the structure functions and various pair correlators give rise the anomalous scaling, i.e., singular dependence on the IR scale with nonlinear anomalous exponents. The latter are determined by the critical dimensions of these operators.

The OPE and the concept of dangerous operators in the stochastic hydrodynamics were introduced and investigated in detail in [39, 40]; see also the review paper [41] and the book [42]. For the rapid-change model, the relationship between the anomalous exponents and dimensions of composite operators was anticipated in [15, 17, 38] within certain phenomenological formulation of the OPE, the so-called "additive fusion rules," typical to the models with multifractal behavior; see also [43, 44]. The RG analysis of Ref. [32] shows that such fusion rules are indeed obeyed by the powers of the local dissipation rate in the model [10], and all these operators are dangerous.

The part of the formal expansion parameter in the RG approach is played by the exponent ζ entering into the velocity correlator; see Eq. (1.9) in Sec. 2 (in Ref. [32], it was denoted by ε, in order to emphasize the analogy with Wilson's ε expansion). The

anomalous exponents were calculated in [32] to the order ζ^2 of the expansion in ζ for any space dimension, and they are in agreement with the first-order results obtained within the zero-mode approach in [14, 15, 16, 17]. In [33], the RG method was generalized to the case of a nonsolenoidal ("compressible") velocity field.

The main advantage of the RG approach (apart from its calculational efficiency) is the universality: it is not related to the aforementioned solvability of the rapid-change model and can equally be applied to the case of finite correlation time, provided the corresponding model possesses the RG symmetry. In [32], the results were presented for the opposite limiting case of the time-independent ("frozen") velocity field.

In this text, we show how to apply the RG and OPE technique to the problem of a passive scalar field advected by a self-similar synthetic Gaussian velocity field with finite correlation time; the steady state is maintained by an imposed linear mean gradient. The velocity field satisfies a linear stochastic equation with effective viscosity and stirring force. The model was proposed and studied in detail (using numerical simulations, in two dimensions) in [3]; its rapid-change version is discussed in [18, 20, 24, 28]. We consider the problem in an arbitrary space dimension, $d \geq 2$; we also stress that the correlation time is not supposed to be small. We establish the existence in the inertial-convective range of several different scaling regimes and show that for some of them the structure functions and other correlation functions of the problem exhibit anomalous scaling behavior; we derive explicit analytical expressions for the corresponding anomalous exponents.

The advection of a passive scalar field in the presence of an imposed linear gradient is described by the equation

$$\nabla_t \theta = \nu_0 \partial^2 \theta - \mathbf{h} \cdot \mathbf{v}, \quad \nabla_t \equiv \partial_t + v_i \, \partial_i. \tag{1.1}$$

Here $\theta(x) \equiv \theta(t, \mathbf{x})$ is the random (fluctuation) part of the total scalar field $\Theta(x) = \theta(x) + \mathbf{h} \cdot \mathbf{x}$, $\mathbf{h}$ is a constant vector that determines distinguished direction, ν_0 is the molecular diffusivity coefficient, $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\partial^2 \equiv \partial_i \partial_i$ is the Laplace operator, and $\mathbf{v}(x) = \{v_i(x)\}$ is the transverse (owing to the incompressibility) velocity field.

The velocity obeys the linear stochastic equation, cf. [3]

$$\partial_t v_i + R v_i = f_i, \tag{1.2}$$

where R [in the momentum representation $R = R(k)$] is a linear operation to be specified below and f_i is an external random stirring force with zero mean and the correlator

$$\langle f_i(x) f_j(x') \rangle = \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) \times$$
$$\times D^f(\omega, k) \exp[-\mathrm{i}(t - t') + \mathrm{i}\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]. \tag{1.3}$$

Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |\mathbf{k}|$ is the wavenumber, and d is the dimensionality of the $\mathbf{x}$ space. Following [3], we choose the correlator D^f to be independent of the frequency,

so that Eq. (1.3) contains the delta-function in time. More specific, we choose

$$D^f(\omega,k) = g_0\nu_0^3\,\sigma_k^{4-d-\varepsilon-\eta}, \quad R(k) = u_0\nu_0\,\sigma_k^{2-\eta}, \tag{1.4}$$

where

$$\sigma_k \equiv \sqrt{k^2+m^2}. \tag{1.5}$$

The positive amplitude factors g_0 (a formal small parameter of the ordinary perturbation theory) and u_0 are the analogs of the the coupling constant ("charge") λ_0 in the standard $\lambda_0\phi^4$ model of critical behavior, see, e.g., [45, 46]; in what follows we shall also term these parameters "coupling constants." The exponents ε and η are the analogs of the RG expansion parameter $\varepsilon = 4-d$ in the $\lambda_0\phi^4$ model, and we shall use the traditional term "ε expansion" in our model for the double expansion in the ε–η plane around the origin $\varepsilon = \eta = 0$, with the additional convention that $\varepsilon = O(\eta)$. The infrared (IR) regularization is provided by the integral scale $L \equiv 1/m$; its precise form is not essential. For $k >> m$ the functions (1.4) take on simple powerlike form. Dimensionality considerations show that the charges are related to the characteristic ultraviolet (UV) momentum scale Λ by

$$g_0 \simeq \Lambda^{\varepsilon+\eta}, \quad u_0 \simeq \Lambda^{\eta}. \tag{1.6}$$

From Eqs. (1.2) and (1.3) it follows that $\mathbf{v}(x)$ obeys Gaussian distribution with zero mean and correlator (dropping the transverse projector)

$$D_v(\omega,k) = \frac{D^f(k)}{\omega^2+R^2(k)} = \frac{g_0\nu_0^3\,\sigma_k^{4-d-\varepsilon-\eta}}{\omega^2+[u_0\nu_0\,\sigma_k^{2-\eta}]^2}\,. \tag{1.7}$$

Therefore, the exponent ε describes the inertial-range behavior of the equal-time velocity correlator or, equivalently, the energy spectrum

$$E(k) \simeq k^{d-1}\int d\omega D_v(\omega,k) \simeq (g_0\nu_0^2/u_0)\,k^{1-\varepsilon}, \tag{1.8}$$

cf. [7, 8, 9], where a close family of models for the velocity field has been considered for a strongly anisotropic shear flow. The second exponent, η, is related to the function $R(k)$, the reciprocal of the correlation time at the wavenumber k ($\eta \equiv 2-z$ in the notation of [7, 8, 9, 37, 38]; our exponents are defined so that $\varepsilon = \eta = 0$ correspond to the starting point of the RG expansion). It then follows that $\varepsilon = 8/3$ gives the Kolmogorow "five-thirds law" for the spatial velocity statistics, and $\eta = 4/3$ corresponds to the Kolmogorov frequency.

It was pointed out in [3] that the linear model (1.2) suffers from the lack of Galilean invariance and therefore does not take into account the self-advection of turbulent eddies. It is well known that the different-time correlations of the Eulerian velocity field are not self-similar, as a result of these "sweeping effects," and depend substantially on the integral scale; see, e.g., [47]. Nevertheless, the results of [3] show that the model gives reasonable description of the passive advection in an appropriate frame, where the mean velocity field vanishes. To justify the model (1.2), we also note that we shall be interested preferably in the equal-time, Galilean invariant quantities (structure

functions, correlations of the dissipation rate etc.), which are not affected by the sweeping effects, and we expect that their absence from the model (1.2) is not essential.

We also note that the model contains two special cases that possess some interest on their own. In the limit $u_0 \to \infty$, $g_0' \equiv g_0/u_0^2 = \text{const}$ we arrive at the rapid-change model:

$$D_v(\omega, k) \to g_0' \nu_0 \, (k^2 + m^2)^{-d/2-\zeta/2}, \quad \zeta \equiv \varepsilon - \eta, \tag{1.9}$$

and the limit $u_0 \to 0$, $g_0'' \equiv g_0/u_0 = \text{const}$ corresponds to the case of a frozen velocity field:

$$D_v(\omega, k) \to g_0'' \nu_0^2 \, (k^2 + m^2)^{-d/2+1-\varepsilon/2} \, \pi \, \delta(\omega), \tag{1.10}$$

when the velocity correlator is independent of the time variable $t - t'$ in the t representation. The latter case for $\mathbf{h} = 0$ has a close formal resemblance with the well-known models of the random walks in random environment with long-range correlations; see [48, 49].

In Sec. 2, we give the field theoretic formulation of the problem and discuss some its consequences; we also explain briefly why the ordinary perturbation theory fails to give correct IR behavior for some values of ε and η and establish the relationship between the IR and UV problems. In Sec. 3, we discuss the UV renormalization of the model, derive the RG equations and present the one-loop expressions for the basic RG functions (beta functions and anomalous dimensions). In Sec. 4, the analysis of the scaling behavior is given. Depending on the values of the exponents ε and η entering into the velocity correlator, the model exhibits various types of IR scaling regimes, associated with the IR stable fixed points of the RG equations:

(i) The anomalous scaling behavior with universal (in the above sense) exponents, characteristic of the rapid-change model, takes place for $\eta < \varepsilon < 2\eta$. The anomalous exponents depend on the only exponent ζ entering into Eq. (1.9).

(ii) The anomalous scaling behavior with the universal exponents, characteristic of the model with time-independent (frozen) velocity field, emerges in the region $\varepsilon > 0$, $\varepsilon > 2\eta$. The exponents are determined solely by the equal-time velocity correlator and depend on the only exponent ε entering into Eq. (1.10).

(iii) The intermediate regime with nonuniversal exponents, which depend on the amplitudes entering into the velocity correlator, emerges for $\varepsilon = 2\eta$; the Kolmogorov-type synthetic velocity field [3] and the case of a local turnover exponent [37] correspond to this regime. The nonuniversality of the exponents in this regime is in agreement with the findings of Ref. [37], where the large d limit has been considered. [However, the exponents in our model turn out to be universal in the one-loop approximation].

(iv) The diffusive-type regimes, for which the advection (i.e., the nonlinearity in Eq. (1.1)) can be treated within the ordinary perturbation theory. These regimes take place in the region specified by the inequalities $\eta > 0$, $\eta > \varepsilon$ and $\eta < 0$, $\varepsilon < 0$.

To avoid possible misunderstandings we emphasize that the limits g_0, $u_0 \to 0$ or g_0, $u_0 \to \infty$ are not supposed to be performed in the original correlation function (1.7); the parameters g_0, u_0 are fixed at some finite values. The behavior specific to the models

(1.9), (1.10) arises asymptotically in the regimes (i) and (ii) as a result of the solution of the RG equations, when the "RG flow" approaches the corresponding fixed point. Therefore, we deal with the finite correlation time, and there is no problem with the steady state in the frozen case even in two dimensions. The regions of IR stability of the regimes (i)–(iv) in the ε–η plane, given above, are identified to the first order of the ε expansion, but some of their boundaries are found exactly.

In the regimes (i)–(iii), the correlation functions of the model exhibit anomalous scaling behavior, i.e., singular dependence on the IR scale m with nonlinear "anomalous exponents." Within the RG and OPE approach, the latter are related to the scaling dimensions of the tensor composite operators $\partial\theta\cdots\partial\theta$; these dimensions are calculated explicitly to the first order of the ε expansion (one-loop approximation) in Sec. 5. The inertial-convective-range asymptotic expressions for the structure functions of arbitrary order (even and odd) and the equal-time correlations of the scalar gradients are obtained in Sec. 6 using the OPE.

The results obtained are reviewed in Sec. 7, where we also discuss briefly the passive advection by the non-Gaussian velocity field governed by the nonlinear stochastic NS equation. Our approach is generalized directly to this case, and the explicit expressions for the anomalous exponents can readily be obtained in the first order of the corresponding ε expansion. We also discuss new problems that arise in the NS model beyond the ε expansion.

2. Field theoretic formulation of the model. IR and UV singularities in the perturbation theory

According to the general theorem (see, e.g., Refs. [45, 46]), the stochastic problem (1.1)–(1.3) is equivalent to the field theoretic model of the doubled set of fields $\Phi\equiv\{\theta,\theta',\mathbf{v},\mathbf{v}'\}$ with action functional

$$\begin{aligned} S(\Phi) &= (1/2)\mathbf{v}'D^f\mathbf{v}' + \mathbf{v}'[-\partial_t\mathbf{v} - R\mathbf{v}] + \\ &+\theta'\left[-\partial_t\theta - (\mathbf{v}\boldsymbol{\partial})\theta + \nu_0\partial^2\theta - \mathbf{h}\cdot\mathbf{v}\right]. \end{aligned} \tag{2.1}$$

Here D^f is the correlator (1.4), the required integrations over $x=(t,\mathbf{x})$ and summations over the vector indices in Eq. (2.1) and analogous formulas below are implied.

The formulation (2.1) means that statistical averages of random quantities in the stochastic problem (1.1)–(1.3) coincide with functional averages with the weight $\exp S(\Phi)$, so that generating functionals of total $[G(A)]$ and connected $[W(A)]$ Green functions are represented by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[S(\Phi) + A\Phi] \tag{2.2}$$

with arbitrary sources $A(x)$ in the linear form

$$A\Phi \equiv \int dx[A^\theta(x)\theta(x) + A^{\theta'}(x)\theta'(x) + A_i^{\mathbf{v}}(x)v_i(x) + A_i^{\mathbf{v}'}(x)v_i'(x)]. \tag{2.3}$$

In the following, we shall not be interested in the Green functions involving the auxiliary vector field $\mathbf{v}'$, so that we can set $A^{\mathbf{v}'} = 0$ in Eq. (2.3). It is then convenient to perform the Gaussian integration over $\mathbf{v}'$ in Eq. (2.2) explicitly. We arrive at the field theoretic model of the reduced set of fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ with the action

$$S(\Phi) = \theta' \left[-\partial_t \theta - (\mathbf{v}\boldsymbol{\partial})\theta + \nu_0 \partial^2 \theta - \mathbf{h} \cdot \mathbf{v} \right] - \mathbf{v} D_v^{-1} \mathbf{v}/2. \tag{2.4}$$

The first four terms in Eq. (2.4) represent the Martin–Siggia–Rose-type action for the stochastic problem (1.1) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$ with the correlator D_v from Eq. (1.7).

The model (2.4) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta'(\mathbf{v}\boldsymbol{\partial})\theta = \theta' V_j v_j \theta$ with vertex factor

$$V_j = \mathrm{i} k_j, \tag{2.5}$$

where $\mathbf{k}$ is the momentum flowing into the vertex via the field θ', and the bare propagators (in the momentum-frequency representation)

$$\begin{aligned}
&\langle \theta\theta' \rangle_0 = \langle \theta'\theta \rangle_0^* = (-\mathrm{i}\omega + \nu_0 k^2)^{-1}, \\
&\langle \theta\theta \rangle_0 = \langle \theta\theta' \rangle_0 h_i h_j \langle v_i v_j \rangle_0 \langle \theta'\theta \rangle_0, \\
&\langle \theta v_i \rangle_0 = -\langle \theta\theta' \rangle_0 h_j \langle v_j v_i \rangle_0, \\
&\langle \theta'\theta' \rangle_0 = 0,
\end{aligned} \tag{2.6}$$

where h_i is a component of the vector $\mathbf{h}$ and the bare propagator $\langle v_i v_j \rangle_0$ is given by Eq. (1.7).

The magnitude $h \equiv |\mathbf{h}|$ can be eliminated from the action (2.4) by rescaling of the scalar fields: $\theta \to h\theta$, $\theta' \to \theta'/h$. Therefore, any total or connected Green function of the form $\langle \theta(x_1) \cdots \theta(x_n)\, \theta'(y_1) \cdots \theta'(y_p) \rangle$ contains the factor of h^{n-p}. The parameter h appears in the bare propagators (2.6) only in the numerators. It then follows that the Green functions with $n - p < 0$ vanish identically. On the contrary, the 1-irreducible function $\langle \theta(x_1) \cdots \theta(x_n)\, \theta'(y_1) \cdots \theta'(y_p) \rangle_{1-\mathrm{ir}}$ contains a factor of h^{p-n} and therefore vanishes for $n - p > 0$; this fact will be relevant in the analysis of the renormalizability of the model (see below).

Another important consequence of the representation (2.2), (2.4) is that the large-scale anisotropy persists, through the dependence on $\mathbf{h}$, for all ranges of momenta (including convective and dissipative ranges), and that the dimensionless ratios of the structure functions are strictly independent on h; cf. [3, 4, 5, 34, 35, 36]. It is noteworthy that all these statements equally hold for any statistics of the velocity field (not necessarily Gaussian or synthetic), provided its distribution is independent of $\mathbf{h}$.

However, the ordinary perturbation theory fails to give correct IR behavior of Green functions for some values of the exponents ε and η. This can easily be illustrated on the simplest example of the 1-irreducible Green function $\langle \theta'\theta \rangle_{1-\mathrm{ir}}$. It satisfies the Dyson equation of the form

$$\langle \theta'\theta \rangle_{1-\mathrm{ir}} = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{\theta'\theta}(\omega, k), \tag{2.7}$$

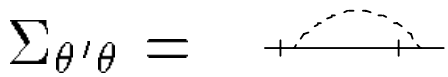

Figure 1. One-loop approximation of the self-energy operator in model (2.1).

where $\Sigma_{\theta'\theta}$ is the self-energy operator represented by the corresponding 1-irreducible diagrams. Its one-loop approximation is shown in Figure 1. There and below the solid lines in the diagrams denote the bare propagator $\langle\theta\theta'\rangle_0$ from Eq. (2.6), the end with a slash corresponds to the field θ', and the end without a slash corresponds to θ; the dashed lines denote the bare propagator (1.7); the vertices correspond to the factor (2.5). The analytic expression for the diagram in Fig. 1 has the form

$$\Sigma_{\theta'\theta}(\omega,k) = -k_i k_j \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{P_{ij}(\mathbf{q})\, D_v(\omega',q)}{-\mathrm{i}(\omega+\omega') + \nu_0(\mathbf{q}+\mathbf{k})^2}\,, \tag{2.8}$$

where $q \equiv |\mathbf{q}|$ and $D(\omega',q)$ is given by Eq. (1.7); the factor of $k_i k_j$ arises from the vertex factors (2.5). Integration over ω' in Eq. (2.8) yields

$$\Sigma_{\theta'\theta}(\omega,k) = -k_i k_j \frac{g_0\nu_0^2}{2u_0} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{P_{ij}(\mathbf{q})\, \sigma_q^{2-d-\varepsilon}}{-\mathrm{i}\omega + \nu_0(\mathbf{q}+\mathbf{k})^2 + u_0\nu_0\sigma_q^{2-\eta}}\,. \tag{2.9}$$

We are interested in the IR behavior of the function (2.9), i.e., the behavior of small k, ω and m. It is easily seen that this behavior is nontrivial in the region on the ε–η plane, determined by the inequalities $\eta < 0$, $\varepsilon > 0$ and $\eta > 0$, $\varepsilon > \eta$, because the integral in (2.9) is then IR divergent if k, ω and m are simply set equal to zero. On the contrary, for the rest of the ε–η plane, the leading term of the desired asymptotic behavior is indeed obtained simply by setting $k = \omega = m = 0$. The analysis is extended directly to the higher-order diagrams; it shows that these IR singularities enhance as the order of a diagram increases, and that they take place only within the same region on the ε–η plane. The IR singularities compensate the smallness of the coupling constant g_0, assumed within the framework of the ordinary perturbation theory. Therefore, in order to find correct IR behavior we would have to sum the entire series even if the expansion parameter, g_0, were small.

It is also clear that these IR singularities get weaker as the parameters ε, η decrease, and they would disappear at $\varepsilon = \eta = 0$ if we could take this limit in Eq. (2.9). However, this is impossible owing to the UV divergence in the integral (2.9) at this point. In general, the diagrams of $\Sigma_{\theta'\theta}$ are UV divergent in the region $\eta > 0$, $\varepsilon < 0$ and $\eta < 0$, $\varepsilon < \eta$, and the UV cutoff at $q \equiv |\mathbf{q}| \simeq \Lambda$ is then implied in (2.9) and higher-order diagrams. If the point $\varepsilon = \eta = 0$ is approached from inside the region of UV convergence, the UV singularities manifest themselves as poles in ε, η and their linear combinations. The elimination of these poles is the classical UV problem, and its solution is given by the standard theory of UV renormalization; the RG equations are obtained within the framework of this theory and express the simple idea of nonuniqueness of the renormalization procedure. The correlation between the IR and UV singularities near the "logarithmic point" $\varepsilon = \eta = 0$, noted above, explains to some extent why the RG

method, which is closely related to the UV divergences, can be a useful tool in studying the IR behavior, and why the exponents ε and η are expected to be relevant small parameters in the RG expansions.

Surprisingly, simple arguments given above lead to reasonable conclusions: the rigorous RG analysis confirms that the Green functions of the model indeed show anomalous IR behavior for some values of ε and η, and the region determined by the inequalities $\eta < 0$, $\varepsilon > 0$ and $\eta > 0$, $\varepsilon > \eta$ coincides with the region of stability of the corresponding fixed points in the *linear* approximation; see Sec. 3, 4.

3. UV renormalization of the model. RG functions and RG equations

The renormalization of the model (2.4) is similar to the renormalization of the simpler rapid-change model, considered in detail in [32]; below we confine ourselves to only the necessary information.

The analysis of UV divergences is based on the analysis of *canonical dimensions*, see [46, 51]. Dynamical models of the type (2.4), in contrast to static models, are two-scale [41, 42, 52], i.e., the action functional (2.4) is invariant with respect to the two independent scale transformations, $S(\Phi', z_i') = S(\Phi, z_i)$, where $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ and $z_i = \{g_0, u_0, \nu_0, m\}$ is the full set of the model parameters. In the first transformation, the time variable is fixed and the space variable is dilated along with all the fields and parameters:

$$\Phi(t, \mathbf{x}) \to \Phi'(t, \mathbf{x}) = \lambda^{d_\Phi^k} \Phi(t, \lambda \mathbf{x}), \quad z_i \to z_i' = \lambda^{d_{z_i}^k} z_i, \tag{3.1}$$

and in the second the space variable is fixed and all the other quantities are dilated:

$$\Phi(t, \mathbf{x}) \to \Phi'(t, \mathbf{x}) = \lambda^{d_\Phi^\omega} \Phi(\lambda t, \mathbf{x}), \quad z_i \to z_i' = \lambda^{d_{z_i}^\omega} z_i. \tag{3.2}$$

Here $\lambda > 0$ is an arbitrary transformation parameter, and two independent canonical dimensions, the momentum dimension d_F^k and the frequency dimension d_F^ω, are assigned to each quantity F (a field or a parameter in the action functional). These canonical ("engineering") dimensions should not be confused with the exact critical dimensions: the latter are subject to nontrivial calculation, while the former are simply determined from the natural normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless [i.e., be invariant with respect to the transformations (3.1) and (3.2) separately]. Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension [41, 42, 52], which corresponds to the dilatation with fixed value of ν_0 (i.e., zero canonical dimension can be assigned to ν_0). In our model, $\partial_t \propto \nu_0 \partial^2$, so that the total dimension is given by $d_F = d_F^k + 2d_F^\omega$.

In the action (2.4), there are fewer terms than fields and parameters, and the canonical dimensions are not determined unambiguously. This is of course a manifestation of the fact that the "superfluous" parameter $h = |\mathbf{h}|$ can be eliminated from the action; see above. After it has been eliminated (or, equivalently, zero canonical dimensions have been assigned to it), the definite canonical dimensions can be assigned

Table 1. Canonical dimensions of the fields and parameters in the model (2.4).

F	θ	θ'	$\mathbf{v}$	ν, ν_0	m, M, μ, Λ	g_0	u_0	g, u, $\mathbf{h}$
d_F^k	-1	$d+1$	-1	-2	1	$\varepsilon+\eta$	η	0
d_F^ω	0	0	1	1	0	0	0	0
d_F	-1	$d+1$	1	0	1	$\varepsilon+\eta$	η	0

to the other quantities. They are given in Table 1, including the dimensions of renormalized parameters, which will appear later on.

From Table 1 it follows that the model is logarithmic (the both coupling constants g_0 and u_0 are dimensionless) at $\varepsilon = \eta = 0$. This means that the UV divergences in the Green functions have the form of the poles in ε, η, and all their possible linear combinations.

The total dimension d_F plays in the theory of renormalization of dynamical models the same role as does the conventional (momentum) dimension in static problems. The canonical dimensions of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \ldots \Phi \rangle_{1-\mathrm{ir}}$ are given by the relations

$$d_\Gamma^k = d - N_\Phi d_\Phi^k, \qquad d_\Gamma^\omega = 1 - N_\Phi d_\Phi^\omega, \qquad d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi, \tag{3.3}$$

where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of divergences in the problem (2.4) should be based on the following auxiliary considerations; cf. [32, 41, 42]:

(1) All the 1-irreducible Green functions with $N_{\theta'} < N_\theta$ vanish; see Sec. 2.

(2) If for some reason a number of external momenta occur as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number (the Green function requires counterterms only if d'_Γ is a non-negative integer).

In the model (2.4), the derivative ∂ at the vertex $\theta'(\mathbf{v}\boldsymbol{\partial})\theta$ can be moved onto the field θ' by virtue of the transversality of the field $\mathbf{v}$. Therefore, in any 1-irreducible diagram it is always possible to move the derivative onto any of the external "tails" θ or θ', which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'}$. This also means that the fields θ, θ' enter into the counterterms only in the form of the derivatives $\partial\theta$ and $\partial\theta'$.

From the dimensions in Table 1 we find $d_\Gamma = d + 2 - N_{\mathbf{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)(1 - N_{\theta'}) - N_{\mathbf{v}}$. From these expressions it follows that for any d, superficial divergences can exist only in the 1-irreducible functions $\langle \theta'\theta \ldots \theta \rangle_{1-\mathrm{ir}}$ with $N_{\theta'} = 1$ and

arbitrary value of N_θ, for which $d_\Gamma = 1 + N_\theta$, $d'_\Gamma = 0$. However, all the functions with $N_\theta > N_{\theta'}$ vanish (see above) and obviously do not require counterterms. As in the case of the rapid-change model [32, 33], we are left with the only superficially divergent function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$; the corresponding counterterm contains two symbols ∂ and is therefore reduced to $\theta'\partial^2\theta$. The inclusion of this counterterm is reproduced by the multiplicative renormalization of the parameters g_0, u_0, and ν_0 in the action functional (2.4):

$$\nu_0 = \nu Z_\nu, \qquad g_0 = g\mu^{\varepsilon+\eta}\, Z_g, \qquad u_0 = u\mu^\eta\, Z_u, \tag{3.4}$$

where the dimensionless parameters g, u, and ν are the renormalized analogs of the bare parameters, μ is the renormalization mass in the minimal subtraction (MS) scheme, which we always use in practical calculations, and $Z_i = Z_i(g,u)$ are the renormalization constants. They satisfy the identities

$$Z_g = Z_\nu^{-3}, \quad Z_u = Z_\nu^{-1}, \tag{3.5}$$

which result from the absence of the renormalization of the contribution with D_v in the functional (2.4). No renormalization of the fields, the "mass" m, and the vector $\mathbf{h}$ is required, i.e., $Z_\Phi = 1$ for all Φ and $Z_m = Z_h = 1$.

The renormalized action functional has the form

$$S_{\mathrm{ren}}(\Phi) = \theta'\left[-\partial_t\theta - (\mathbf{v}\boldsymbol{\partial})\theta + \nu Z_\nu\partial^2\theta - \mathbf{h}\cdot\mathbf{v}\right] - \mathbf{v}D_v^{-1}\mathbf{v}/2, \tag{3.6}$$

where the correlator D_v is expressed in renormalized parameters using the formulas (3.4):

$$D_v(\omega,k) = \frac{g\nu^3\mu^{\varepsilon+\eta}\sigma_k^{4-d-\varepsilon-\eta}}{\omega^2 + [u\nu\mu^\eta\sigma_k^{2-\eta}]^2}\,. \tag{3.7}$$

The relation $S(\Phi, e_0) = S_{\mathrm{ren}}(\Phi, e, \mu)$ (where e_0 is the complete set of bare parameters, and e is the set of renormalized parameters) for the generating functional $W(A)$ in Eq. (2.2) yields $W(A, e_0) = W_{\mathrm{ren}}(A, e, \mu)$. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG differential equation:

$$\mathcal{D}_{RG}\, W_{\mathrm{ren}}(A, e, \mu) = 0, \tag{3.8}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_g(g,u)\partial_g + \beta_u(g,u)\partial_u - \gamma_\nu(g,u)\mathcal{D}_\nu. \tag{3.9}$$

In Eq. (3.9), we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and the RG functions (the β functions and the anomalous dimension γ) are defined as

$$\gamma_\nu \equiv \widetilde{\mathcal{D}}_\mu \ln Z_\nu, \tag{3.10}$$

$$\beta_g \equiv \widetilde{\mathcal{D}}_\mu g = g[-\varepsilon - \eta + 3\gamma_\nu], \tag{3.11}$$

$$\beta_u \equiv \widetilde{\mathcal{D}}_\mu u = u[-\eta + \gamma_\nu]. \tag{3.12}$$

The relations between β and γ in Eqs. (3.10) and (3.11) result from the definitions and the relation (3.5).

Now let us turn to the explicit calculation of the constant Z_ν in the one-loop approximation in the MS scheme. The constant Z_ν is determined by the requirement that the 1-irreducible Green function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$, when expressed in renormalized variables, be UV finite [i.e., have no singularities for ε, $\eta \to 0$]. The Dyson equation (2.7) relates this function to the self-energy operator $\Sigma_{\theta'\theta}$, and Eq. (2.9) gives the explicit expression for the latter in the first order $O(g_0)$ of the unrenormalized perturbation theory. Now we have to calculate the function $\Sigma_{\theta'\theta}$ in the order $O(g)$ of the renormalized perturbation theory; therefore we should simply replace $\nu_0 \to \nu$ in the propagator $\langle\theta\theta'\rangle_0$ and use the expression (3.7) for the velocity correlator in Fig. 1, which leads to the substitution $g_0 \to g\mu^{\varepsilon+\eta}$, $u_0 \to u\mu^\eta$, $\nu_0 \to \nu$ in Eqs. (2.8), (2.9). We know that the divergent part of the diagram is independent of ω, so that we can set $\omega = 0$ in what follows. It is also convenient to cut off the integral over $\mathbf{q}$ from below at $q \simeq m$ and set $m = 0$ in the integrand (the integral diverges logarithmically, and its UV divergent part is independent of the specific form of the IR regularization). Furthermore, we can set $\mathbf{k} = 0$ in the integrand (we know that the counterterm is proportional to k^2, and the factor of k^2 has already been isolated from the integral) and make use of the isotropy, namely,

$$\int d\mathbf{q}\, f(q) P_{ij}(\mathbf{q}) = \delta_{ij} \frac{d-1}{d} \int d\mathbf{q}\, f(q).$$

Then Eq. (2.9) yields

$$\Sigma_{\theta'\theta}(\omega = 0, k) \simeq -k^2\, \frac{g\nu\mu^\varepsilon\,(d-1)\,J}{2ud}\,, \tag{3.13}$$

where we have written

$$J \equiv \int \frac{d\mathbf{q}}{(2\pi)^d}\, \frac{q^{-d-\varepsilon}}{1 + u\,(\mu/q)^\eta} \tag{3.14}$$

and $\simeq$ denotes the equality up to the UV finite parts. The expansion of the integrand in u gives

$$J = \sum_{s=0}^{\infty} (-u)^s \mu^{s\eta} \int \frac{d\mathbf{q}}{(2\pi)^d} q^{-d-\varepsilon-s\eta} \simeq \frac{S_d}{(2\pi)^d} \sum_{s=0}^{\infty} (-u)^s \frac{\mu^{s\eta}\, m^{-\varepsilon-s\eta}}{\varepsilon + s\eta}\,, \tag{3.15}$$

where the parameter m arises from the IR limit in the integral over $\mathbf{q}$ and $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d-dimensional space.

Finally, from Eqs. (3.13) and (3.15) we obtain

$$\Sigma_{\theta'\theta}(\omega = 0, k) \simeq \frac{-ag\nu k^2}{u} \sum_{s=0}^{\infty} \frac{(-u)^s\,(\mu/m)^{\varepsilon+s\eta}}{\varepsilon + s\eta}, \tag{3.16}$$

where we have written

$$a \equiv \frac{(d-1)S_d}{2d(2\pi)^d}. \tag{3.17}$$

The renormalization constant Z_ν is found from the requirement that the UV divergences cancel out in Eq. (2.7) after the substitution $\nu_0 = \nu Z_\nu$. This determines Z_ν up to an UV finite contribution; the latter is fixed by the choice of the renormalization

scheme. In the MS scheme all the renormalization constants have the form "1 + only poles in ε, η and their linear combinations," which gives the following expression

$$Z_\nu = 1 - \frac{ag}{u}\sum_{s=0}^{\infty}\frac{(-u)^s}{\varepsilon + s\eta}\,, \tag{3.18}$$

with the coefficient a from Eq. (3.17).

In contrast to the rapid-change model, the one-loop approximation in the case at hand is not exact: the expression (3.18) has nontrivial corrections of order g^2, g^3, and so on. The series in Eq. (3.18) can be expressed in the form of a single integral, but this is not convenient for the calculation of the RG functions.

The RG functions in the one-loop approximation can be calculated from the renormalization constant (3.18) using the identity $\widetilde{\mathcal{D}}_\mu = \beta_g\partial_g + \beta_u\partial_u$, which follows from the definitions (3.10), (3.11) and the fact that Z_ν depends only on the charges g, u. Within our accuracy this identity is reduced to $\widetilde{\mathcal{D}}_\mu \simeq -(\varepsilon+\eta)\mathcal{D}_g - \eta\mathcal{D}_u$. From Eq. (3.18) it then follows:

$$\gamma_\nu = \Big[(\varepsilon+\eta)\mathcal{D}_g + \eta\mathcal{D}_u\Big]\frac{ag}{u}\sum_{s=0}^{\infty}\frac{(-u)^s}{\varepsilon+s\eta} = \frac{ag}{u}\sum_{s=0}^{\infty}(-u)^s = \frac{ag}{u(1+u)}\,, \tag{3.19}$$

up to the corrections of order g^2 and higher. The beta functions are obtained from Eq. (3.19) using the relations (3.11), (3.12).

4. Fixed points and scaling regimes

It is well known that possible scaling regimes of a renormalizable model are associated with the IR stable fixed points of the corresponding RG equations, see, e.g., [45, 46]. The fixed points are determined from the requirement that all the beta functions of the model vanish. In our model the coordinates g_*, u_* of the fixed points are found from the equations

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0 \tag{4.1}$$

with the beta functions given in Eqs. (3.11), (3.12). The type of the fixed point is determined by the eigenvalues of the matrix $\Omega = \{\Omega_{ik} = \partial\beta_i/\partial g_i\}$, where β_i denotes the full set of the beta functions and g_i is the full set of charges. For the standard (as in Eq. (1.6)) formulation of the problem the IR asymptotic behavior is governed by the IR stable fixed points, i.e., those for which all the eigenvalues are positive.

From the equations (3.11), (3.12) we obtain the exact relation $\beta_g/g - 3\beta_u/u = 2\eta - \varepsilon$. It shows that the beta functions β_g, β_u cannot vanish simultaneously for finite values of their arguments. [The only exception is the case $2\eta = \varepsilon$. We shall study it separately, and for now we assume $2\eta \neq \varepsilon$.] Therefore, to find the fixed points we must set either $u = 0$ or $u = \infty$ and simultaneously rescale g so that the anomalous dimension γ_ν remain finite.

In order to study the limit $u \to \infty$ we change to the new variables $w \equiv 1/u$, $g' \equiv g/u^2$; the corresponding beta functions have the form

$$\beta_w \equiv \widetilde{\mathcal{D}}_\mu w = -\beta_u/u^2 = w[\eta - \gamma_\nu],$$
$$\beta_{g'} \equiv \widetilde{\mathcal{D}}_\mu g' = \beta_g/u^2 - 2g\beta_u/u^3 = g'[\eta - \varepsilon + \gamma_\nu], \tag{4.2}$$

and for the one-loop anomalous dimension we obtain from Eq. (3.19)

$$\gamma_\nu = ag'/(1+w) \tag{4.3}$$

with the constant a defined in Eq. (3.17). From the expressions (4.2) we find two fixed points, which we denote FPI and FPII. The first point is trivial,

$$\text{FPI}: \qquad w_* = g'_* = 0; \qquad \gamma^*_\nu = 0. \tag{4.4}$$

The corresponding matrix Ω is diagonal with the diagonal elements

$$\Omega_1 = \eta, \quad \Omega_2 = \eta - \varepsilon. \tag{4.5}$$

For the second point we obtain

$$\text{FPII}: \qquad w_* = 0,\ g'_* = (\varepsilon - \eta)/a; \qquad \gamma^*_\nu = \varepsilon - \eta. \tag{4.6}$$

The corresponding matrix Ω is triangular, $\partial_{g'}\beta_w = 0$, and its eigenvalues coincide with the diagonal elements:

$$\Omega_1 = \partial_w \beta_w = \eta - \gamma^*_\nu = 2\eta - \varepsilon,$$
$$\Omega_2 = \partial_{g'}\beta_{g'} = ag'_* = \varepsilon - \eta. \tag{4.7}$$

We note that the expressions for γ^*_ν in Eq. (4.6) and for Ω_1 in (4.7) are exact, i.e., they have no corrections of order $O(\varepsilon^2)$ [we take $\varepsilon \simeq \eta$, so that here and below $O(\varepsilon^2)$ denotes all the terms of the form $\varepsilon\eta$, η^2 and higher].

Now let us turn to the regime with $u \to 0$. In order to study this limit we change to the new variable $g'' \equiv g/u$; the corresponding beta functions have the form

$$\beta_{g''} \equiv \widetilde{\mathcal{D}}_\mu g'' = \beta_g/u - g\beta_u/u^2 = g''[-\varepsilon + 2\gamma_\nu],$$
$$\beta_u = u[-\eta + \gamma_\nu] \tag{4.8}$$

[the function β_u is the same as in Eq. (3.12)]. The one-loop anomalous dimension (3.19) takes the form

$$\gamma_\nu = ag''/(1+u). \tag{4.9}$$

From the expressions (4.8) we find two fixed points, which we denote FPIII and FPIV. The first point is trivial,

$$\text{FPIII}: \qquad u_* = g''_* = 0; \qquad \gamma^*_\nu = 0. \tag{4.10}$$

The corresponding matrix Ω is diagonal with the elements

$$\Omega_1 = -\varepsilon, \quad \Omega_2 = -\eta. \tag{4.11}$$

For the nontrivial point we obtain

$$\text{FPIV}: \qquad u_* = 0,\ g''_* = \varepsilon/2a; \qquad \gamma^*_\nu = \varepsilon/2. \tag{4.12}$$

The corresponding matrix Ω is triangular, $\partial_{g''}\beta_u = 0$, and its eigenvalues have the form

$$\Omega_1 = \partial_u \beta_u = -\eta + \gamma_\nu^* = (\varepsilon - 2\eta)/2,$$
$$\Omega_2 = \partial_{g''}\beta_{g''} = 2ag_*'' = \varepsilon. \tag{4.13}$$

The expressions for γ_ν^* in Eq. (4.12) and for Ω_1 in Eq. (4.13) are exact. Of course, the expressions (4.5), (4.11), and $\gamma_\nu^* = 0$ for the trivial fixed points are also exact.

In the special case $\varepsilon = 2\eta$ the beta functions (3.11), (3.12) become proportional, and the set (4.1) reduces to a single equation. As a result, the corresponding nontrivial fixed point, which we denote FPV, is degenerate: rather than a point, we have a line of fixed points in the g–u plane. It is given by the relation

$$\text{FPV}: \qquad g_*/u_*(u_*+1) = \eta/a; \qquad \gamma_\nu^* = \eta = \varepsilon/2. \tag{4.14}$$

The exact expression for γ_ν^* follows from the relation between the RG functions in Eqs. (3.10) and (3.11). The eigenvalues of the matrix Ω (which is not diagonal here) have the form

$$\Omega_1 = 0, \quad \Omega_2 = \eta\,(2+u_*)/(1+u_*). \tag{4.15}$$

The vanishing of the element Ω_1 reflects the existence of a marginal direction in the g–u plane (along the line of the fixed points) and is therefore an exact fact. The coordinates of a point on the line (4.14) can also be expressed explicitly as functions of the dimensionless parameter $\rho \equiv g_0/u_0^3$ using the exact relation $g_0/u_0^3 = g_*/u_*^3$. The actual expansion parameter appears to be $\sqrt{\eta}$ rather than η itself, and the zeroth order approximation has the form

$$g_* = (\eta/a)^{3/2}\rho^{-1/2}, \qquad u_* = (\eta/a\rho)^{1/2}, \qquad \Omega_2 = 2\eta. \tag{4.16}$$

In Figure 2, we show the regions of stability for the fixed points FPI–FPV in the ε–η plane, i.e., the regions for which the eigenvalues of the Ω matrix are positive. The boundaries of the regions are depicted by thick lines. We note that the regions adjoin each other without overlaps or gaps. This fact is exact for the ray $\varepsilon = 2\eta > 0$, the boundary between the regions of stability for the points FPII and FPIV [at the same time, this ray is the region of stability for the point FPV]. On the contrary, the boundary $\varepsilon = \eta$, $\eta > 0$ for the point FPII and $\varepsilon = 0$, $\varepsilon > \eta$ for FPIV are approximate, so that the gaps or overlaps can appear in the two-loop approximation. The regions denoted as FPIVa and FPIVb with the boundary $\varepsilon = 2$ both correspond to the same fixed point FPIV; the part FPIVb represents the region in which the velocity field has negative critical dimension.

Surprisingly, Fig. 2 has some resemblance with the phase diagrams presented in Refs. [7, 9], despite the essential difference between the models (in those papers, a strongly anisotropic velocity field has been studied). Indeed, the boundaries between the diffusive-type behavior ("homogenization regime" in terminology of [7]) and convective-type regimes ("superdiffusive behavior") in the two models coincide (however, in our case they are not exact and will be affected by the $O(\varepsilon)$ corrections). Furthermore, the Kolmogorov point ($\varepsilon = 8/3$, $\eta = 4/3$) in our case and in [7] lies on a boundary

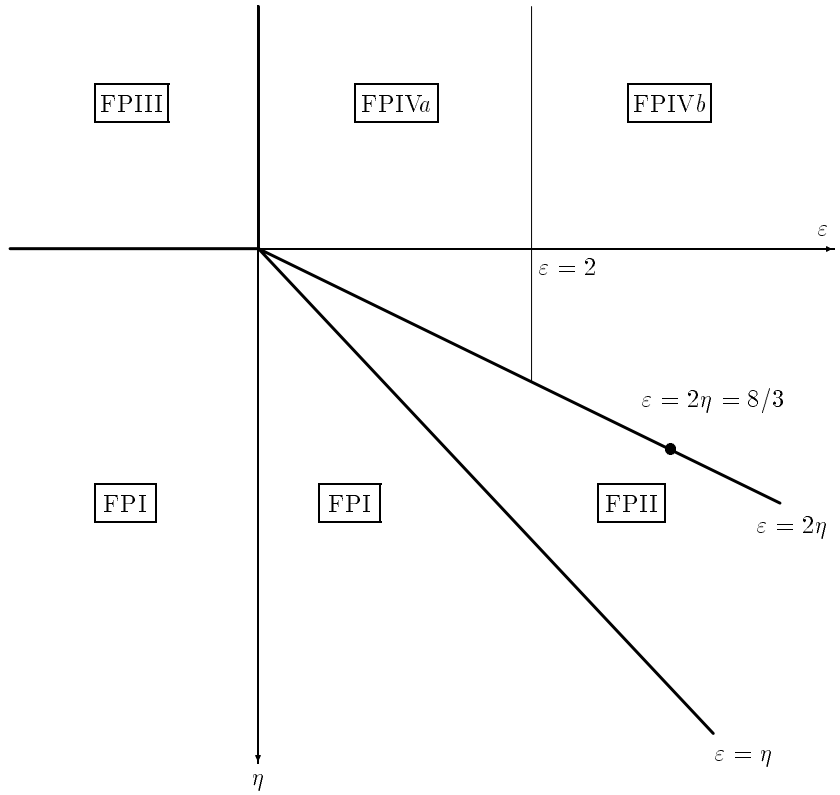

Figure 2. Regions of stability of the fixed points in model (2.1).

between two nontrivial regimes. We also note that the boundary $2\eta = \varepsilon$ between the rapid-change and frozen regimes was anticipated on phenomenological grounds in Ref. [37], see also [38]; their arguments can be linked directly to the RG analysis (see below).

It is clear from the definition of the parameters g', g'' that the critical regime governed by the point FPII corresponds to the rapid-change limit (1.9) of our model, while the point FPIV corresponds to the limit of the frozen velocity field; see Eq. (1.10). This shows that in the latter case, the temporal fluctuations of the velocity field are asymptotically irrelevant in determining the inertial-range behavior of the scalar, which is then completely determined by the equal-time velocity statistics. In the former case, spatial and temporal fluctuations are both relevant, but the effective correlation time of the scalar field becomes so large under renormalization that the correlation time of the velocity can be completely neglected. The inertial-range behavior of the scalar is determined solely by the $\omega = 0$ mode of the velocity field; this is the case of the rapid-change model.

We then expect that all the critical dimensions at the point FPII [FPIV] depend on the only exponent $\zeta \equiv \varepsilon - \eta$ [ε] that survives in the limit in question, and coincide with the corresponding dimensions obtained directly for the models (1.9) [(1.10)]. This is indeed the case; see Eqs. (4.20) and (5.13) below.

In the regimes governed by the trivial fixed points FPI and FPIII, the contribution of the convection dies out in the IR asymptotic region; the IR behavior has purely diffusive character, while the convection can be treated within ordinary perturbation theory. The existence of the two fixed points, the frozen and the rapid-change ones, implies that for $\eta < 0$ transport by *small* wavenumbers $k \to 0$ is governed by equal-time (spatial) velocity statistics, while for $\eta > 0$ transport by small wavenumbers is determined by the $\omega = 0$ mode, i.e., the time decorrelated component of the velocity field. If there were IR singularities in the scalar correlations, they would be determined by the contributions of small momenta, and these two regimes would be really different. However, in the regions of stability of the trivial fixed points there are no such singularities (see the discussion in Sec. 2). Moreover, in these regimes all momenta k contribute to the long-term, large-scale transport properties of the scalar field (we recall that for $\eta > 0$, $\varepsilon < 0$ and $\eta < 0$, $\varepsilon < \eta$, the actual UV cutoff Λ has to be introduced, see Sec. 2, and the main contribution to the perturbative diagrams then comes from the momenta of order $k \sim \Lambda$). The RG is not suitable for studying such "Λ divergent," analytic in momenta and frequecies, quantities. Therefore, the splitting of the homogenization regime into the rapid-change and frozen parts is not meaningless, but not practically useful. Probably for this reason it was not mentioned in Refs. [7, 8, 9]. In what follows, we shall focus our attention on the nontrivial (anomalous) regimes.

The solution of the RG equations in conformity with the stochastic hydrodynamics is discussed in Refs. [40, 41, 42] in detail; see also [32, 33] for the case of the rapid-change models. Below we restrict ourselves with the only information we need.

Any solution of the RG equation (3.8) can be represented in terms of invariant variables $\bar{g}(k)$, $\bar{u}(k)$, and $\bar{\nu}(k)$, i.e., the first integrals normalized at $k = \mu$ to g, u, and ν, respectively (we recall that μ is the renormalization mass in the MS scheme). The relation between the bare and invariant charges has the form

$$g_0 = k^{\varepsilon+\eta}\, \bar{g}\, Z_g(\bar{g}, \bar{u}), \quad u_0 = k^{\eta}\, \bar{u}\, Z_u(\bar{g}, \bar{u}), \quad \nu_0 = \bar{\nu} Z_\nu(\bar{g}, \bar{u}), \tag{4.17}$$

see, e.g., [41, 42, 50]. Equation (4.17) determines implicitly the invariant variables as functions of the bare parameters; it is valid because both sides of it satisfy the RG equation, and because Eq. (4.17) at $k = \mu$ coincides with (3.4) owing to the normalization of the invariant variables.

Correlation time of the velocity field at the wavenumber k is determined by the relation $t_v^{-1}(k) = R(k) = u_0 \nu_0 k^{2-\eta}$, see Eqs. (1.4), (1.7). Correlation time of the free scalar field is given by $t_\theta^{-1}(k) = \nu_0 k^2$, in the presence of advection it is replaced by the exact expression $t_\theta^{-1}(k) = \bar{\nu}(k) k^2$. The relations (3.5) and (4.17) allow the bare parameters and renormalization constants to be eliminated from the ratio $t_\theta(k)/t_v(k)$; this gives

$$t_\theta(k)/t_v(k) = \bar{u}(k) \propto \text{const}\ k^{-\eta+\gamma_\nu^*}. \tag{4.18}$$

The last relation in Eq. (4.18) holds for $k \to 0$. It follows from the RG equation $\mathcal{D}_k \bar{u} = \beta_u(\bar{g}, \bar{u})$, which reduces to $\mathcal{D}_k \bar{u} = \bar{u}[-\eta + \gamma_\nu^*]$ near a fixed point; see Eq.

(3.12). Equation (4.18) discloses the precise physical meaning of the invariant variable $\bar{u}$: the ratio of the velocity and scalar correlation times at the wavenumber k. Now we can complete the above discussion of the scaling regimes and relate it to the phenomenological arguments given in Refs. [37, 38]. From (4.18) it follows that for the fixed points FPI and FPII the velocity correlation time $t_v(k)$ becomes very small in comparison to $t_\theta(k)$ for $k \to 0$ and can be disregarded; we arrive at the time-decorrelated velocity field. For FPIII and FPIV, the opposite inequality, $t_v(k) >> t_\theta(k)$, holds for small momenta, the temporal fluctuations of the velocity are "frozen in," and its correlation time can be replaced with $t_v(k) = \infty$. [Using the representation (4.18) and the exact expressions for γ_ν^* in Eqs. (4.4), (4.6), (4.10) and (4.12), one can easily check that $\bar{u} \to \infty$ for FPI and FPII and $\bar{u} \to 0$ for FPIII and FPIV, in agreement with the analysis of the Ω matrix.] However, these strong inequalities for the correlation times hold only asymptotically for $k \to 0$, and therefore the exact correlator (1.4) can be replaced with its limits (1.9) or (1.10) only in calculation of a quantity dominated by small k modes of the velocity field. Finally, for the point FPV one has $\gamma_\nu^* = \eta$ and the ratio (4.18) remains finite for $k \to 0$; this is the case of the local turnover exponent, studied in [37].

Let F be some multiplicatively renormalized quantity (a parameter, a field or composite operator), i.e., $F = Z_F F_{\rm ren}$ with certain renormalization constant Z_F. Then its critical dimension is given by the expression

$$\Delta[F] \equiv \Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \tag{4.19}$$

see, e.g., [40, 41, 42, 52]. Here d_F^k and d_F^ω are the corresponding canonical dimensions, γ_F^* is the value of the anomalous dimension $\gamma_F(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F$ at the fixed point in question, and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of frequency. For the nontrivial fixed points we obtain

$$\Delta_\omega = 2 - \begin{cases} \zeta & \text{for FPII,} \\ \varepsilon/2 & \text{for FPIV,} \\ \eta = \varepsilon/2 & \text{for FPV} \end{cases} \tag{4.20}$$

(we recall that $\zeta \equiv \varepsilon - \eta$, see (1.9)). The critical dimensions of the fields Φ in our model are also found exactly:

$$\Delta_{\mathbf{v}} = 1 - \gamma_\nu^*, \qquad \Delta_\theta = -1 \qquad \Delta_{\theta'} = d + 1, \tag{4.21}$$

and for the IR scale we have $\Delta_m = 1$ [we recall that all these quantities in the model (2.4) are not renormalized and therefore their anomalous dimensions vanish identically, $\gamma_{\Phi,m} \equiv 0$]. It is also not too difficult to show that the composite operator θ^n in the model (2.4) is not renormalized and therefore its critical dimension is given simply by the relation $\Delta[\theta^n] = n\Delta[\theta]$; cf. [32] for the rapid-change case.

We note that the canonical dimensions of the fields θ, θ' in our model (see Table I) differ from their counterparts in the isotropic rapid-change model (see Table I in Ref. [32]). As a result, the critical dimensions (4.20) and (4.21) at the point FPIV differ from their analogs for the rapid-change model, in spite of the fact that the anomalous

dimensions are identical. In principle, the canonical dimensions in two models can be made equal by an appropriate rescaling of the scalar fields; we shall not dwell on this point here.

Let $G(r) = \langle F_1(x)F_2(x')\rangle$ be an equal-time two-point quantity, for example, the pair correlation function of the primary fields Φ or some multiplicatively renormalizable composite operators. The existence of a nontrivial IR stable fixed point implies that in the IR asymptotic region $\Lambda r >> 1$ and any fixed mr the function $G(r)$ takes on the form

$$G(r) \simeq \nu_0^{d_G^\omega}\, \Lambda^{d_G} (\Lambda r)^{-\Delta_G}\, \xi(mr), \tag{4.22}$$

with the values of the critical dimensions that correspond to the fixed point in question and certain scaling function ξ whose explicit form is not determined by the RG equation itself. The canonical dimensions d_G^ω, d_G and the critical dimension Δ_G ot the function $G(r)$ are equal to the sums of the corresponding dimensions of the quantities F_i.

5. Critical dimensions of the composite operators $\partial\theta \cdots \partial\theta$

In the following, an important role will be played by the composite operators of the form

$$F[n,p] \equiv \partial_{i_1}\theta \cdots \partial_{i_p}\theta\, (\partial_i\theta\partial_i\theta)^l, \tag{5.1}$$

where p is the number of the free vector indices and $n = p+2l$ is the total number of the fields θ entering into the operator; the vector indices of the symbol $F[n,p]$ are omitted.

Coincidence of the field arguments in Green functions containing a composite operator F gives rise to additional UV divergences. They are removed by a special renormalization procedure, described in detail, e.g., in [45, 46, 51]. The discussion of the renormalization of composite operators in turbulence models can be found in [41, 42]; see also Ref. [32] for the case of Kraichnan's model. Owing to the renormalization, the critical dimension $\Delta[F]$ associated with certain operator F is not in general equal to the simple sum of critical dimensions of the fields and derivatives entering into F. As a rule, the renormalization of composite operators involves mixing, i.e., an UV finite renormalized operator is a linear combination of unrenormalized operators, and vice versa.

The analysis of UV divergences is related to the analysis of the corresponding canonical dimensions, cf. Sec. 3. It shows that the operators $F[n,p]$ mix only with each other in renormalization, with the multiplicative matrix renormalization of the form (dropping the vector indices everywhere)

$$F[n,p] = Z_{[n,p]\,[n',p']} F_{\rm ren}[n',p'] \tag{5.2}$$

Here $F_{\rm ren}$ is the renormalized analog of the operator F and Z is the matrix of renormalization constants. The corresponding matrix of anomalous dimensions is defined as

$$\gamma_{[n,p]\,[n',p']} = Z^{-1}_{[n,p]\,[n'',p'']} \widetilde{\mathcal{D}}_\mu Z_{[n'',p'']\,[n',p']}. \tag{5.3}$$

$$\Gamma_n = F[n,p] + \frac{1}{2}$$

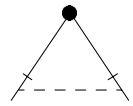

Figure 3. One-loop approximation of the function (5.4) in model (2.1).

A simple analysis of the diagrams shows that the matrix element $Z_{[n,p]\,[n',p']}$ is proportional to $h^{n-n'}$, so that the elements with $n < n'$ vanish (the parameter $h \equiv |\mathbf{h}|$ appears only in the numerators of the diagrams; see Sec. 3). The elements with $n = n'$ are independent of h and therefore they can be calculated directly in the isotropic model with $\mathbf{h} = 0$. The block $Z_{[n,p]\,[n,p']}$ can be then diagonalized by the changing to irreducible operators (scalars, vectors, and traceless tensors); but for our purposes it is sufficient to note that the elements $Z_{[n,p]\,[n,p']}$ vanish for $p < p'$ [the irreducible tensor of the rank p consists of the monomials with $p' \leq p$ only, and therefore only these monomials can admix to the monomial of the rank p in renormalization]. Therefore, the renormalization matrix in Eq. (5.2) is triangular, and so is the matrix (5.3). The isotropy is violated for $\mathbf{h} \neq 0$, so that the irreducible tensors with different numbers of the fields θ can mix with each other even though their ranks are also different. In particular, the vector $\partial_i\theta$ admixes to the irreducible tensor $\partial_i\theta\partial_j\theta - \delta_{ij}(\partial_s\theta\partial_s\theta)/d$ in the form of the traceless combination $2\delta_{ij}(h_s\partial_s\theta)/d - h_i\partial_j\theta - h_j\partial_i\theta$. In the following, we shall not be interested in the precise form of the basis operators, i.e., those having definite anomalous dimensions; we shall rather be interested in the anomalous dimensions themselves. The latter are given by the eigenvalues $\gamma[n,p]$ of the matrix (5.3), and in our case they are completely determined by the diagonal elements of the renormalization matrix (5.2): $\gamma[n,p] = \widetilde{\mathcal{D}}_\mu \ln Z_{[n,p]\,[n,p]}$.

Now let us turn to the one-loop calculation of the constant (5.11) in the MS scheme. Let $\Gamma(x;\theta)$ be the generating functional of the 1-irreducible Green functions with one composite operator $F[n,p]$ from Eq. (5.1) and any number of fields θ. Here $x \equiv (t,\mathbf{x})$ is the argument of the operator and $\theta(x)$ is the functional argument, the "classical analog" of the random field θ. We are interested in the θ^n term of the expansion of $\Gamma(x;\theta)$ in $\theta(x)$, which we denote $\Gamma_n(x;\theta)$; it has the form

$$\Gamma_n(x;\theta) = \frac{1}{n!}\int dx_1 \cdots \int dx_n\, \theta(x_1)\cdots\theta(x_n) \times$$
$$\times \langle F[n,p](x)\theta(x_1)\cdots\theta(x_n)\rangle_{1-\mathrm{ir}}. \tag{5.4}$$

In the one-loop approximation the function (5.4) is represented diagramatically as shown in Fig. 3. The first term is the "tree" approximation, and the black circle with two attached lines in the diagram denotes the variational derivative $\delta^2 F[n,p]/\delta\theta\delta\theta$. In the momentum representation it has the form

$$T(\mathbf{k},\mathbf{q}) \equiv \frac{\delta^2 F[n,p]}{\delta\theta(\mathbf{k})\delta\theta(\mathbf{q})} = -p\,(p-1)\,k_{i_1}q_{i_2}\,(\partial_{i_3}\theta\cdots\partial_{i_p}\theta)\,(\partial_i\theta\partial_i\theta)^l$$
$$-4pl\,k_{i_1}q_s\,(\partial_{i_2}\theta\cdots\partial_{i_p}\theta)\,\partial_s\theta\,(\partial_i\theta\partial_i\theta)^{l-1}$$

$$-2l(\mathbf{k}\cdot\mathbf{q})\,(\partial_{i_1}\theta\cdots\partial_{i_p}\theta)\,(\partial_i\theta\partial_i\theta)^{l-1}$$
$$-4l(l-1)\,k_jq_s(\partial_j\theta\partial_s\theta)\,(\partial_{i_1}\theta\cdots\partial_{i_p}\theta)\,(\partial_i\theta\partial_i\theta)^{l-2}. \tag{5.5}$$

Strictly speaking, we had to symmetrize the right-hand side of Eq. (5.5) with respect to the indices $i_1\cdots i_p$ and the momenta $\mathbf{k}$, $\mathbf{q}$. However, the symmetry is restored automatically after the vertex $T(\mathbf{k},\mathbf{q})$ has been inserted into the diagram, which is why only one term of each type is displayed in Eq. (5.5) and the required symmetry coefficients are introduced.

The vertex (5.5) contains $(n-2)$ factors of $\partial\theta$. Two remaining "tails" θ are attached to the vertices $\theta'(\mathbf{v}\boldsymbol{\partial})\theta$ of the diagram in Fig. 3. It follows from the explicit form of the vertices that these two fields θ are isolated from the diagram in the form of the overall factor $\partial\theta\partial\theta$; cf. Sec. 3. In other words, two external momenta, corresponding to these fields θ, occur as an overall factor in the diagram, and the UV divergence of the latter is logarithmic rather than quadratic; cf. the expression (2.9), (3.13). Therefore, we can set all the external momenta and the "mass" m equal to zero in the integrand; the IR regularization is provided by the cut-off of the integral at $q\simeq m$. Then the UV divergent part of the one-loop diagram in Fig. 3 can be written in the form

$$\partial_p\theta\partial_l\theta\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{q}}{(2\pi)^d}\,T(\mathbf{q},-\mathbf{q})\frac{P_{pl}(\mathbf{q})\,D_v(\omega,q)}{\omega^2+\nu^2q^4}. \tag{5.6}$$

The expression (5.6) is a linear combination of the integrals

$$T_{ij,pl}=\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{q}}{(2\pi)^d}\,\frac{q_iq_j\,P_{pl}(\mathbf{q})\,D_v(\omega,q)}{\omega^2+\nu^2q^4}. \tag{5.7}$$

We perform the integration over ω and make use of the isotropy, namely,

$$\int d\mathbf{q}\,f(q)\,q_iq_j\,P_{pl}(\mathbf{q})=\frac{(d+1)\delta_{pl}\delta_{ij}-\delta_{pi}\delta_{lj}-\delta_{pj}\delta_{li}}{d(d+2)}\int d\mathbf{q}\,f(q)\,q^2.$$

This gives

$$T_{ij,pl}=\frac{(d+1)\delta_{pl}\delta_{ij}-\delta_{pi}\delta_{lj}-\delta_{pj}\delta_{li}}{2ud(d+2)}\,g\mu^{\varepsilon}\,J \tag{5.8}$$

with the integral J from Eq. (3.14).

Substituting Eqs. (5.5) and (5.8) into Eq. (5.6) gives the desired expression for the divergent part of the diagram in Fig. 3. In this expression we have to take into account all the terms proportional to the operator $F[n,p]$ and neglect all the other terms, namely, the terms containing the factors of $\delta_{i_1i_2}$ etc. The latter determine non-diagonal elements of the matrix (5.2), which we are not interested in here. Finally we obtain

$$\Gamma_n\simeq F[n,p]\left[1-\frac{g\mu^{\varepsilon}\,J\,Q[n,p]}{4ud(d+2)}\right]+\cdots, \tag{5.9}$$

where we have written

$$Q_{np}\equiv 2n\,(n-1)-(d+1)\,(n-p)\,(d+n+p-2)=$$
$$=2p\,(p-1)-(d-1)\,(n-p)\,(d+n+p). \tag{5.10}$$

The dots in Eq. (5.9) stand for the $O(g^2)$ terms and the structures different from $F[n,p]$, $\simeq$ denotes the equality up to the UV finite parts; we also recall that $n=p+2l$.

The constant $Z_{[n,p],[n,p]}$ is found from the requirement that the renormalized analog $\Gamma_n^{\rm ren} \equiv Z^{-1}_{[n,p],[n,p]}\Gamma_n$ of the function (5.9) be UV finite (mind the minus sign in the exponent); along with the representation (3.15) for the integral J and the MS scheme this gives the following result:

$$Z_{[n,p]\,[n,p]} = 1 - \frac{ag}{u}\,\frac{Q[n,p]}{2(d-1)(d+2)}\sum_{s=0}^{\infty}\frac{(-u)^s}{\varepsilon+s\eta}\,, \tag{5.11}$$

with the polynomial $Q[n,p]$ from Eq. (5.10) and the constant a is given in Eq. (3.17).

For the anomalous dimension (5.3) we then obtain:

$$\gamma[n,p] = \frac{ag\,Q[n,p]}{2u(u+1)(d-1)(d+2)}\,; \tag{5.12}$$

cf. Sec. 3 for the dimension γ_ν. The critical dimension associated with the operator $F[n,p]$ has the form $\Delta[n,p] = \gamma^*[n,p]$; see Eq. (4.19) and Table I (γ^* denotes the value of γ at the fixed point in question). For the nontrivial fixed points discussed in Sec. 4 we then obtain

$$\Delta[n,p] = \frac{Q[n,p]}{2(d-1)(d+2)} \times \begin{cases} \zeta \equiv \varepsilon - \eta & \text{for FPII,} \\ \varepsilon/2 & \text{for FPIV,} \\ \eta = \varepsilon/2 & \text{for FPV} \end{cases} \tag{5.13}$$

with the corrections of order $O(\varepsilon^2)$.

The expression (5.13) illustrates the general fact that the critical dimensions in the rapid-change and frozen regimes depend only on the exponents ζ and ε, respectively. It turns out that the dimension $\Delta[n,p]$ at the point FPV is universal, i.e., it is independent of the free parameter u_*, or, equivalently, of the specific choice of a fixed point on the curve described by Eq. (4.14). This is a consequence of the explicit form of the RG functions in the one-loop approximation (the same combination $g/u(u+1)$ enters into the beta functions and the anomalous dimension of the operator $F[n,p]$). We then expect that the exact dimension $\Delta[n,p]$ at the point FPV is nonuniversal, and the dependence on u_* will appear at the two-loop level. Another artifact of the one-loop approximation is the continuity of the dimension $\Delta[n,p]$ at the crossover line $\varepsilon = 2\eta$ as a function of the exponents ε, η.

The first-order result (5.13) for the operator $F[2,0]$ (the local dissipation rate) is in fact exact. The proof is based on certain Schwinger equation; it is almost identical to the analogous proof for the Kraichnan model, given in [32], and will not be discussed here.

The above analysis applies also to the case of a nonsolenoidal velocity field (compressible fluid). The transversal projector in Eq. (1.3) is then replaced with $P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})$, where $Q_{ij}(\mathbf{k}) \equiv k_i k_j/k^2$ is the longitudinal projector and $\alpha > 0$ is an additional arbitrary parameter, the degree of compressibility. For the rapid-change regime (1.9), the dimension $\Delta[n,p]$ takes on the form

$$\Delta[n,p] = \frac{-\zeta}{(d+2)}\left[\frac{(n-p)(d+n+p)}{2} + \right.$$

$$+\frac{p(p-1)(\alpha-1)+\alpha(n-p)(n+p-2)}{(d-1+\alpha)}\Bigg]+O(\zeta^2), \tag{5.14}$$

in agreement with the $p=0$ results obtained in Refs. [31] for the 'tracer' model and earlier in Ref. [25] for $d=1$. In general case (1.7), additional superficial UV divergence emerges in the 1-irreducible Green function $\langle\theta'\theta v\rangle_{1-\mathrm{ir}}$, and the second independent renormalization constant should be introduced as a coefficient in front of the new counterterm $\theta'(\mathbf{v}\boldsymbol{\partial})\theta$. This case requires special analysis and will be discussed elsewhere (in particular, the nontrivial fixed point becomes infinite for the purely potential frozen velocity field, cf. [48, 49] for the random walks in random environment).

6. Operator product expansion and the anomalous scaling for the structure functions and other correlators

The representation (4.22) for any scaling function $\xi(mr)$ describes the behavior of the Green function for $\Lambda r >> 1$ and any fixed value of mr. The inertial-convective range corresponds to the additional condition that $mr << 1$. The form of the function $\xi(mr)$ is not determined by the RG equations themselves; in the theory of critical phenomena, its behavior for $mr \to 0$ is studied using the well-known Wilson operator product expansion (OPE); see, e.g., [45, 46, 51]. This technique is also applicable to the theory of turbulence; see [32, 33, 39, 40, 41, 42].

According to the OPE, the equal-time product $F_1(x)F_2(x')$ of two renormalized operators at $\mathbf{x}\equiv(\mathbf{x}+\mathbf{x}')/2=\mathrm{const}$ and $\mathbf{r}\equiv\mathbf{x}-\mathbf{x}'\to 0$ has the representation

$$F_1(x)F_2(x')=\sum_\alpha C_\alpha(\mathbf{r})F_\alpha(\mathbf{x},\mathbf{t}), \tag{6.1}$$

where the functions C_α are the Wilson coefficients regular in m^2 and F_α are all possible renormalized local composite operators allowed by symmetry, with definite critical dimensions Δ_α. The renormalized correlator $\langle F_1(x)F_2(x')\rangle$ is obtained by averaging Eq. (6.1) with the weight $\exp S_{ren}$, the quantities $\langle F_\alpha\rangle$ appear on the right-hand side. Their asymptotic behavior for $m\to 0$ is found from the corresponding RG equations and has the form $\langle F_\alpha\rangle\propto m^{\Delta_\alpha}$. From the operator product expansion (6.1) we therefore find the following expression for the scaling function $\xi(mr)$ in the representation (4.22) for the correlator $\langle F_1(x)F_2(x')\rangle$:

$$\xi(mr)=\sum_\alpha A_\alpha\,(mr)^{\Delta_\alpha}, \tag{6.2}$$

where the coefficients $A_\alpha=A_\alpha(mr)$ are regular in $(mr)^2$.

In the models of critical phenomena, the leading contribution to the representations like (6.2) is related to the simplest operator $F=1$ with the minimal dimension $\Delta_\alpha=0$, while the other operators determine only the corrections that vanish for $mr\to 0$. The feature characteristic of the turbulence models is the existence of the so-called "dangerous" composite operators with negative critical dimensions [32, 33, 39, 40, 41, 42]. Their contributions to the operator product expansions determine

the IR behavior of the scaling functions and lead to their singular dependence on m for $mr \to 0$.

If the spectrum of the dimensions Δ_α for a given scaling function is bounded from below, the leading term of its behavior for $mr \to 0$ is simply given by the minimal dimension. This is the case of the rapid-change model (see [32, 33]), and, as we shall see below, of our model (2.4). [The exception is provided by the non-rapid-changes regimes, if the values of the exponents ε, η are large enough. It is discussed in the subsequent Section.]

Consider for definiteness the equal-time structure functions of the scalar field:

$$S_n(r) \equiv \langle [\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^n \rangle, \quad r \equiv |\mathbf{x} - \mathbf{x}'|. \tag{6.3}$$

For the functions (6.3), the representation of the form (4.22) is valid with the dimensions $d_G^\omega = 0$ and $d_G = \Delta_G = n\Delta_\theta = -n$. In general, the operators entering into the operator product expansions are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. The leading term of the Taylor expansion for the function (6.3) is given by the n-th rank tensor $F[n,n]$ from Eq. (5.1). The decomposition of $F[n,n]$ in irreducible tensors gives rise to the dimensions $\Delta[n,p]$ with all possible values of p; the admixture of junior operators gives rise to all the dimensions $\Delta[k,p]$ with $k < n$. Therefore, the asymptotic expression for the structure function has the form

$$S_n(r) \simeq (hr)^n \sum_{k=0}^{n} \sum_{p=p_k}^{k} \left[C_{kp}\, (mr)^{\Delta[k,p]} + \cdots \right]. \tag{6.4}$$

Here and below p_k denotes the minimal possible value of p for given k, i.e., $p_k = 0$ for k even and $p_k = 1$ for k odd; C_{kp} are some numerical coefficients dependent on ε, η, d, and on the angle between the vectors $\mathbf{h}$ and $\mathbf{r}$.

Some remarks are in order.

The dots in Eq. (6.4) stand for the contributions of the order $(mr)^{2+O(\varepsilon)}$ and higher, which arise from the senior operators, for example, $\partial^2\theta\partial^2\theta$ or $\mathbf{v}^2$.

In the original Kraichnan model, only scalar operators give contributions to the representations like (6.4), because the mean values $\langle F_\alpha \rangle$ of all the other irreducible tensors vanish owing to the isotropy; see [32, 33]. In the model (2.4), the traceless irreducible tensors acquire nonzero mean values, and their dimensions appear on the right-hand side of Eq. (6.4). In particular, the mean value of the operator $\partial_i\theta\partial_j\theta - \delta_{ij}(\partial_s\theta\partial_s\theta)/d$ is proportional to the traceless tensor of the form $\delta_{ij}(h_s h_s)/d - h_i h_j$, its tensor indices are contracted with the indices of the corresponding coefficient C_α in Eq. (6.1).

The operators $F[k,p]$ with $k > n$ (whose contributions would be more important) do not appear in Eq. (6.4), because they do not appear in the Taylor expansion of the function S_n and do not admix in renormalization to the terms of the Taylor expansion.

The leading term of the expression (6.4) for $mr \to 0$ is obviously given by the contribution with the minimal possible dimension. The straightforward analysis of the explicit one-loop expression (5.13) shows that for fixed n, any $d \geq 2$, and any

nontrivial fixed point, the dimension $\Delta[n,p]$ decreases monotonically with p and reaches its minimum for the minimal possible value of $p = p_n$, i.e., $p = 0$ if n is even and $p = 1$ if n is odd. Furthermore, this minimal value $\Delta[n,p_n]$ decreases monotonically as n increases, i.e.,

$$\Delta[2k,0] > \Delta[2k+1,1] > \Delta[2k+2,0].$$

[A similar hierarchy has been established recently in Ref. [53] for the magnetic field advected passively by the rapid-change velocity in the presence of large-scale anisotropy.] Therefore, the desired leading term for the even (odd) structure function S_n is determined by the scalar (vector) composite operator consisting of n factors $\partial\theta$ and has the form

$$S_n(r) \propto (hr)^n \, (mr)^{\Delta[n,p_n]} \tag{6.5}$$

with the dimension $\Delta[n,p]$ given in Eq. (5.13).

For the rapid-change fixed point and even values of n, the total power of r in Eq. (6.5) coincides with the exponent in the original isotropic Kraichnan model, calculated to the order $O(\zeta)$ in [17] and $O(1/d)$ in [15] within the zero-mode approach, and to the order $O(\zeta^2)$ in [32] within the framework of the RG. We also note that the anomalous dimensions associated with the operators $F[2k,2]$ were calculated in [32] to the order $O(\zeta^2)$; the exact dimension of the operator $F[2,2]$ was found in [14]. [It should be noted that the *decomposition* of the total exponent in Eq. (6.5) into the critical dimension of the composite operator and the critical dimension of the structure function itself differs from the analogous decomposition for Kraichnan's model, as a result of the difference in canonical dimensions; see Sec. 3].

The result (6.5) for the third-order structure function in the rapid-change model coincides with the $O(\zeta)$ result obtained in [28] within the zero-mode technique; see also the earlier paper [18] for the three-dimensional result. We note that the exponents $-7\zeta/5$ and $3\zeta/5$ from [18] should be identified with the anomalous dimensions $\Delta[3,1]$ and $\Delta[3,3]$, respectively. The result (6.5) for $n = 3$ is also in agreement with the $O(1/d)$ result obtained in [19], with the identification $\gamma + 1 - \Delta = 3 + \Delta[3,1]$.

For the case of the frozen velocity field (FPIV), the first-order results for the even structure functions were presented in [32]. We also note that they satisfy the exact inequalities obtained for the time-independent case in [38].

The analysis given above is extended directly to the case of other correlation functions. For example, the analog of the expression (6.5) for the equal-time pair correlation function of the operators (5.1) has the form

$$\langle F[n,p]\, F[n',p'] \rangle \simeq h^{n+n'} \, (\Lambda r)^{-\Delta[n,p_n]-\Delta[n',p_{n'}]} (mr)^{\Delta[n+n',\, p_{n+n'}]} \, . \tag{6.6}$$

Some special cases of the relation (6.6) for the rapid-change model were obtained earlier in Refs. [14, 15, 16, 17, 32].

Another interesting example is the equal-time pair correlator $\langle \theta^n(t,\mathbf{x}) \theta^k(t,\mathbf{x}') \rangle$. Substituting the relations $d_G^\omega = 0$ and $d_G = \Delta_G = -(n+k)$ into the general expression (4.22) gives $\langle \theta^n \theta^k \rangle = r^{n+k} \xi(mr)$, and the small mr behavior of the scaling function $\xi(mr)$

is found from Eq. (6.2) (here and below, we do not display the obvious dependence on h). In contrast to the previous example, composite operators in the expansion (6.1) can involve the field θ *without derivatives*. The leading term in Eq. (6.2) is then given simply by the operator θ^{n+k} with $\Delta_F = -(n+k)$, while the first correction is related to the monomial $(\partial_i\theta\partial_i\theta)\theta^{n+k-2}$ whose critical dimension is easily found to be $\Delta_F = -(n+k) + \Delta_\omega$ with Δ_ω from Eq. (4.20). Therefore, in the inertial range our correlator has the form $\langle\theta^n\theta^k\rangle \simeq c_1 m^{-(n+k)} - c_2 m^{-(n+k)}(mr)^{\Delta_\omega} + \ldots$, a large constant minus a powerlike correction (the signs of the constants c_i are explained by the fact that the correlator is positive and decreases as r grows). In the structure functions (6.3) all the contributions related to operators containing fields without derivatives cancel out to give the behavior (6.4), determined by the operators constructed only of field derivatives.

Finally, we note that the hierarchy of critical dimensions $\Delta[n,p]$, established in Sec.5, persists also for the nonsolenoidal velocity field, see (5.14). Therefore, the asymptotic expressions like (6.4), (6.5) and (6.6) remain valid for the compressible case (a 'tracer' in terminology of [31]) with the exponents $\Delta[n,p]$ given in (5.14).

7. Conclusion

We have applied the RG and OPE methods to the simple model describing the advection of a passive scalar by the synthetic velocity field and in the presence of an imposed linear mean gradient. The statistics of the velocity is Gaussian, with a given self-similar correlator with finite correlation time.

We have shown that the model possesses the RG symmetry, and the corresponding RG equations have several fixed points. As a result, the correlation functions of the scalar field in the inertial-convective range exhibit various types of scaling behavior: diffusive-type regimes for which the advection can be treated within the ordinary perturbation theory, and three nontrivial convection-type regimes for which the correlation functions of the model reveal anomalous scaling behavior. The stability of the fixed points (and, therefore, the choice of the scaling regime) depends on the values of the two exponents ε and η, entering into the velocity correlator.

The explicit asymptotic expressions for the structure functions and other correlation functions in any space dimension are obtained; the anomalous exponents are calculated to the first order of the corresponding ε expansions. For the first nontrivial regime the anomalous exponents are the same as in the rapid-change version of the model; for the second they are the same as in the model with time-independent (frozen) velocity field. In these regimes, the anomalous exponents are universal in the sense that they depend only on the exponents entering into the velocity correlator; what is more, they depend on the only exponent ($\zeta \equiv \varepsilon - \eta$ and ε) remaining in the corresponding limit. For the last regime the exponents are nonuniversal: in principle, they depend also on the values of the coupling constants. It turns out, however, that they can reveal the nonuniversality only in the second order of the ε expansion.

A serious question is that of the validity of the ε expansion and the possibility of the extrapolation of the results, obtained within the ε expansions, to the finite values $\varepsilon = O(1)$. In Refs. [22] and [28], the agreement between the nonperturbative results and the ε expansion has been established on the example of the triple correlation function in the rapid-change model. In particular, in [28] the exponent $\Delta[3,1]$ (we use the notation introduced in Sec. 5) has been calculated numerically for all $0 \le \zeta \le 2$ within the zero-mode approach. It was shown that for small ζ, this nonperturbative result agrees with the expansion in ζ, while for $\zeta = 2$ it coincides with the exact analytic result $\Delta[3,1] = -2$ obtained previously in [20]. In the paper [25], the one-dimensional version of the rapid-change model has been studied both numerically and analytically within the zero-mode approach; the analytic expressions for the anomalous exponents obtained within the ε expansion have also been found to agree with the nonperturbative numerical results. Finally, in Ref. [29] the analytic $O(\varepsilon)$ result has been confirmed by the numerical experiment on the example of the fourth-order structure function in three dimensions.

In this connection, we also note that a number of exact analytic results appear to be in agreement with the corresponding ε expansions: the exponent $\Delta[2,2]$, calculated exactly in [14], the exponent for the second-order structure function of a passively advected magnetic field [27], and the second-order exponent for a scalar advected by the nonsolenoidal ("compressible") velocity field [33]; the corresponding expansions in ζ (to the order ζ^2) have been calculated within the RG approach in [32, 33]. These facts support strongly the applicability of the ε expansion, at least for low-order correlation functions.

In the paper [11], a closure-type approximation for the rapid-change model, the so-called linear ansatz, was used to derive simple explicit expression for the anomalous exponents for any $0 \le \zeta \le 2$, d, and n, the order of the structure function. For $\zeta = 1$, the predictions of the linear ansatz appear to be consistent with the numerical simulations [12, 23, 29, 30]; they are also in agreement with some exact relations [13, 21, 23]. However, they do not agree with the results obtained within the zero-mode and RG approaches in the ranges of small ζ, $2-\zeta$ or $1/d$. In fact, there is no *formal* contradiction between the perturbative results and the linear ansatz: the violation of the latter in the aforementioned limits can be related to the fact that they have strongly nonlocal dynamics in the momentum space; see, e.g., the discussion in Refs. [13, 29]. On the other hand, the *numerical* divergence of the predictions given by the linear ansatz and ε expansion for the fourth-order structure function at $\zeta \simeq 1$ is, roughly speaking, of the same order of magnitude as the difference between the nonperturbative numerical results and the perturbative small-ζ results for the triple correlator, as one can see from the figures presented in [22]. One can think that the series in ζ, obtained within the RG or zero-mode approaches, give correct formal expansions of the (unknown) exact exponents, while the linear ansatz gives a good approximate expression for the same quantities near $\zeta = 1$. We also note that the numerical agreement between the expansion in ζ and the exact results is expected to worsen as n increases, because the

actual expansion parameter is $n\zeta$ rather than ζ itself; see [32, 33].

Let us conclude with a brief discussion of the passive advection in the non-Gaussian velocity field governed by the nonlinear stochastic NS equation. In this case, one has to add the nonlinear term $(v_j\partial_j)v_i$ to the left-hand side of the equation (1.2) and set $\eta = 0$ in Eq. (1.4). The RG approach is also applicable to this model; the analysis of the UV divergences shows that the basic RG functions are the same as for the model with $\mathbf{h} = 0$. The RG analysis of the latter has been accomplished in [55]. It shows that the model possesses a nontrivial IR stable fixed point; its coordinate has been calculated in [55] in the first order of the ε expansion. The inclusion of a nonzero mean gradient $\mathbf{h} \neq 0$ gives rise to anomalous scaling; the analysis given in Sec. 6 can also be extended to this case. For small ε, the anomalous exponents are given by the relation (5.12), in which one should take $g/u(u+1) = \varepsilon/3a$ at the fixed point, with the coefficient a from Eq. (3.17) (we use the notation introduced above; the definition of the parameters ε, a, and u in [55] is slightly different). Despite the non-Gaussianity, the critical dimensions of the powers of the velocity field are given by the simple linear relation $\Delta[v_{i_1}\cdots v_{i_n}] = n\Delta_{\mathbf{v}} = n(1-\varepsilon/3)$; see [39, 40, 41, 42, 56] (in the notation of the papers [39, 40, 41, 42], ε should be replaced with 2ε). Therefore, all these operators are dangerous for $\varepsilon > 3$, and the summation of their contributions is required. For the different-time correlators, it has been accomplished in [39, 40]; for the structure functions it can be performed in the one-loop approximation and leads to an analogous conclusion: the behavior of the second-order structure function does not change for $\varepsilon > 3$. For $\varepsilon > 4$, the composite operator of the local energy dissipation rate also becomes dangerous [52], possibly along with all of its powers [56]; some other dangerous operators arise for $\varepsilon > 6$ and further [54, 57]. The identification of all the other dangerous operators and summation of their contributions in the operator product expansions remains an open problem.

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