САНКТ-ПЕТЕРБУРГСКИЙ
ГОСУДАРСТВЕННЫЙ
УНИВЕРСИТЕТ

ПРИОРИТЕТНЫЙ
НАЦИОНАЛЬНЫЙ ПРОЕКТ
"ОБРАЗОВАНИЕ"

Проект «Инновационная образовательная среда в классическом университете»

Пилотный проект № 22 «Разработка и внедрение
инновационной образовательной программы «Прикладные математика и физика»»

Физический факультет

Кафедра физики высоких энергий и элементарных частиц

А.Н.Васильев, Н.В.Антонов

ANOMALOUS SCALING IN TURBULENCE II.
Anisotropy and compressibility

Учебно-методическое пособие

Санкт Петербург
2007 г.

- Рецензент: профессор кафедры статистической физики физического факультета СПбГУ, д.ф.-м.н. Л.Ц.Аджемян.

- Печатается по решению методической комиссии физического факультета СПбГУ.

- Рекомендовано Ученым советом физического факультета СПбГУ.

Anomalous scaling in turbulence II. Anisotropy and compressibility. – СПб., 2007

В учебно-методическом пособии с помощью квантово-полевых методов рассмотрены процессы турбулентной конвекции. Подробно обсуждается влияние анизотропии и сжимаемости.
Пособие предназначено для студентов 4-7-го курсов, аспирантов, соискателей и других обучающихся по специальности теоретическая физика.

Abstract. The field theoretic renormalization group and operator product expansion are applied to the problem of a passive scalar advected by a Gaussian nonsolenoidal velocity field with finite correlation time, in the presence of large-scale anisotropy. The energy spectrum of the velocity in the inertial range has the form $E(k) \propto k^{1-\varepsilon}$, and the correlation time at the wavenumber k scales as $k^{-2+\eta}$. It is shown that, depending on the values of the exponents ε and η, the model exhibits various types of inertial-range scaling regimes with nontrivial anomalous exponents. Explicit asymptotic expressions for the structure functions and other correlation functions are obtained; they are represented by superpositions of power laws with nonuniversal amplitudes and universal (independent of the anisotropy) anomalous exponents, calculated to the first order in ε and η in any space dimension. These anomalous exponents are determined by the critical dimensions of tensor composite operators built of the scalar gradients, and exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution to the inertial-range behaviour. The leading terms of the even (odd) structure functions are given by the scalar (vector) operators. The anomalous exponents depend explicitly on the degree of compressibility.

1. Introduction

The investigation of intermittency and anomalous scaling in fully developed turbulence remains essentially an open theoretical problem. Both the natural and numerical experiments suggest that the deviation from the predictions of the classical Kolmogorov–Obukhov theory [1, 2] is even more strongly pronounced for a passively advected scalar field than for the velocity field itself; see, e.g., [3, 4] and literature cited therein. At the same time, the problem of passive advection appears to be easier tractable theoretically: even simplified models describing the advection by a "synthetic" velocity field with a given Gaussian statistics reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Therefore, the problem of passive scalar advection, being of practical importance in itself, may also be viewed as a starting point in studying anomalous scaling in the turbulence on the whole.

The most progress has been achieved for the so-called rapid-change model, introduced by Kraichnan [5]: for the first time, the anomalous exponents have been calculated on the basis of a microscopic model and within regular perturbation expansions; see, e.g., [6, 7, 8, 9, 10] and references therein.

In the original Kraichnan model, the velocity field is taken to be Gaussian, isotropic, incompressible and decorrelated in time. More realistic models should involve anisotropy, compressibility and finite correlation time. The recent studies have pointed up some significant differences between the zero and finite correlation-time problems [11, 12, 13] and between the compressible and incompressible cases [14, 15, 16, 17, 18]. It is noteworthy that the nonsolenoidal velocity field remains nontrivial in the one dimensional case, which is more accessible to numerical simulations and allows interesting comparison between the numerical and analytical results; see [19].

Another important question recently addressed is the effects of large-scale anisotropy on inertial-range statistics of passively advected fields [3, 4, 8, 9, 13, 20, 21, 22] and the velocity itself [23]. These studies have shown that the anisotropy present at large scales has a strong influence on the small-scale statistical properties of the scalar, in disagreement with what was expected on the basis of the cascade ideas [3, 4, 8, 9, 22]. On the other hand, the exponents describing the inertial-range scaling exhibit universality and hierarchy related to the degree of anisotropy, which gives some quantitative support to Kolmogorov's hypothesis on the restored local isotropy of the inertial-range turbulence [13, 20, 23].

In this paper, we apply the field theoretic renormalization group (RG) and operator product expansion (OPE) to the problem of a passive scalar advected by a Gaussian self-similar nonsolenoidal velocity field with finite correlation time and in the presence of an imposed linear mean gradient.

Detailed exposition of the RG and OPE approach to statistical models of fully developed turbulence and the bibliography can be found in [24, 25]. The incompressible version of the model in question is discussed in [13] in detail; below we concentrate on the features specific to the nonsolenoidal velocity field.

The plan of the paper is the following. In Sec. 2, we describe the model and give its field theoretic formulation. In Sec. 3, we discuss the ultraviolet (UV) renormalization of the model, derive the RG equations and calculate the basic RG functions (beta functions and anomalous dimensions) in the one-loop approximation. The RG equations possess three nontrivial infrared (IR) stable fixed points, which establishes the existence of three different IR scaling regimes (Sec. 4). The solution of the RG equations for an equal-time two-point correlation function is given, which determines its dependence on the UV scale. In Sec. 5, we present explicit asymptotic expressions for the structure functions of the scalar field and other correlation functions and discuss the hierarchy of the anomalous exponents associated with the anisotropic contributions. They are determined by the critical dimensions of tensor composite operators built of the scalar gradients. In Sec. 6, we briefly discuss an alternative model of the passive advection by the compressible fluid and present respective inertial-range expressions.

2. Description of the model. The field theoretic formulation

The advection of a passive scalar field in the presence of an imposed linear gradient is described by the equation

$$\partial_t \theta + (\mathbf{v}\cdot\boldsymbol{\partial})\,\theta = \nu_0 \partial^2\theta - \mathbf{h}\cdot\mathbf{v}. \tag{2.1}$$

Here $\theta(x) \equiv \theta(t,\mathbf{x})$ is the random (fluctuation) part of the total scalar field $\Theta(x) = \theta(x) + \mathbf{h}\cdot\mathbf{x}$, $\mathbf{h}$ is a constant vector that determines distinguished direction, ν_0 is the molecular diffusivity coefficient, $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, and $\partial^2 \equiv \partial_i\partial_i$ is the Laplace operator. The velocity field $\mathbf{v}(x) = \{v_i(x)\}$ obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x)v_j(x')\rangle = \int \frac{d\omega}{2\pi}\int \frac{d\mathbf{k}}{(2\pi)^d}\Big\{P_{ij}(\mathbf{k})+\alpha Q_{ij}(\mathbf{k})\Big\}\times$$
$$\times D_v(\omega,k)\exp[-\mathrm{i}(t-t')+\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]. \tag{2.2}$$

Here $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ and $Q_{ij}(\mathbf{k})=k_ik_j/k^2$ are the transverse and the longitudinal projectors, respectively, $k\equiv|\mathbf{k}|$, and d is the dimensionality of the $\mathbf{x}$ space, $\alpha>0$ is a free parameter. For the function D_v we choose

$$D_v(\omega,k)=\frac{g_0u_0\nu_0^3\,k^{4-d-\varepsilon-\eta}}{\omega^2+[u_0\nu_0\,k^{2-\eta}]^2}\,. \tag{2.3}$$

For the energy spectrum we then obtain $E(k)\simeq k^{d-1}\int d\omega D_v(\omega,k)\simeq g_0\nu_0^2\,k^{1-\varepsilon}$. Therefore, the coupling constant $g_0>0$ and the exponent ε describe the equal-time velocity correlator or, equivalently, the energy spectrum, while the constant $u_0>0$ and the exponent η are related to the frequency $\omega\simeq u_0\nu_0\,k^{2-\eta}$, characteristic of the mode k. The exponents ε and η are the analogs of the RG expansion parameter $\varepsilon=4-d$ in the theory of critical behaviour, and we shall use the traditional term "ε expansion" for the double expansion in the ε–η plane around the origin $\varepsilon=\eta=0$, with the additional convention that $\varepsilon=O(\eta)$. The IR regularization is provided by the cut-off in the integral (2.2) from below at $k\simeq m$, where $\ell\equiv 1/m$ is the outer turbulence scale. Dimensionality considerations show that the coupling constants g_0, u_0 are related to the characteristic UV momentum scale Λ by

$$g_0\simeq\Lambda^{\varepsilon},\quad u_0\simeq\Lambda^{\eta}. \tag{2.4}$$

The model contains two special cases that possess some interest on their own. In the limit $u_0\to\infty$, $g'_0\equiv g_0/u_0=\text{const}$ we arrive at the rapid-change model:

$$D_v(\omega,k)\to g'_0\nu_0\,k^{-d-\zeta},\quad \zeta\equiv\varepsilon-\eta, \tag{2.5}$$

and the limit $u_0\to 0$, $g_0=\text{const}$ corresponds to the case of a "frozen" velocity field:

$$D_v(\omega,k)\to g_0\nu_0^2\,k^{-d+2-\varepsilon}\,\pi\,\delta(\omega), \tag{2.6}$$

when the velocity correlator is independent of the time variable $t-t'$ in the t representation.

The stochastic problem (2.1), (2.2) is equivalent to the field theoretic model of the set of three fields $\Phi\equiv\{\theta,\theta',\mathbf{v}\}$ with action functional

$$S(\Phi)=\theta'\left[-\partial_t\theta-(\mathbf{v}\cdot\boldsymbol{\partial})\,\theta+\nu_0\partial^2\theta-\mathbf{h}\cdot\mathbf{v}\right]-\mathbf{v}D_v^{-1}\mathbf{v}/2. \tag{2.7}$$

The first four terms in Eq. (2.7) represent the Martin–Siggia–Rose-type action (see, e.g., Refs. [25, 26]) for the stochastic problem (2.1) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$. Here D_v is the correlator (2.2), the required integrations over $x=(t,\mathbf{x})$ and summations over the vector indices are implied.

The formulation (2.7) means that statistical averages of random quantities in the stochastic problem (2.1), (2.2) coincide with functional averages with the weight

$\exp S(\Phi)$, so that generating functionals of total $[G(A)]$ and connected $[W(A)]$ Green functions are represented by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[S(\Phi) + A\Phi] \tag{2.8}$$

with arbitrary sources $A(x)$ in the linear form

$$A\Phi \equiv \int dx\, [A^{\theta}(x)\theta(x) + A^{\theta'}(x)\theta'(x) + A_i^{\mathbf{v}}(x)v_i(x)]. \tag{2.9}$$

The model (2.7) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta'(\mathbf{v}\cdot\boldsymbol{\partial})\theta = \theta' V_j v_j \theta$ with vertex factor $V_j = -\mathrm{i}k_j$, where $\mathbf{k}$ is the momentum flowing into the vertex via the field θ, and the bare propagators

$$\begin{aligned} &\langle\theta\theta'\rangle_0 = \langle\theta'\theta\rangle_0^* = (-\mathrm{i}\omega + \nu_0 k^2)^{-1}, \quad \langle\theta\theta\rangle_0 = \langle\theta\theta'\rangle_0 h_i \langle v_i v_j\rangle_0 h_j \langle\theta'\theta\rangle_0, \\ &\langle\theta v_i\rangle_0 = -\langle\theta\theta'\rangle_0 h_j \langle v_j v_i\rangle_0, \quad \langle\theta'\theta'\rangle_0 = 0, \end{aligned} \tag{2.10}$$

where h_i is a component of the vector $\mathbf{h}$ and the bare propagator $\langle v_i v_j\rangle_0$ is given by Eq. (2.3).

The magnitude $h \equiv |\mathbf{h}|$ can be eliminated from the action (2.7) by rescaling of the scalar fields: $\theta \to h\theta$, $\theta' \to \theta'/h$. Therefore, any total or connected Green function of the form $\langle\theta(x_1)\cdots\theta(x_n)\,\theta'(y_1)\cdots\theta'(y_p)\rangle$ contains the factor of h^{n-p}. The parameter h appears in the bare propagators (2.10) only in the numerators. It then follows that the Green functions with $n-p<0$ vanish identically. On the contrary, the 1-irreducible function $\langle\theta(x_1)\cdots\theta(x_n)\,\theta'(y_1)\cdots\theta'(y_p)\rangle_{1-\mathrm{ir}}$ contains the factor h^{p-n} and therefore vanishes for $n-p>0$; this fact will be relevant in the next Section in the analysis of the renormalizability of the model.

3. UV renormalization. RG functions and RG equations

The analysis of UV divergences is based on the analysis of canonical dimensions. Dynamical models of the type (2.7), in contrast to static models, have two scales, i.e., the canonical dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω. They are determined so that $[F] \sim [L]^{-d_F^k}[T]^{-d_F^\omega}$, where L is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_t \propto \partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems.

In the action (2.7), there are fewer terms than fields and parameters, and the canonical dimensions are not determined unambiguously. This is of course a manifestation of the fact that the "superfluous" parameter $h = |\mathbf{h}|$ can be eliminated from the action; see Sec. 2. After it has been eliminated (or, equivalently, zero

Table 1. Canonical dimensions of the fields and parameters in the model (2.7).

F	θ	θ'	$\mathbf{v}$	ν, ν_0	m, μ, Λ	g_0	u_0	g, u, $\mathbf{h}$, α
d_F^k	-1	$d+1$	-1	-2	1	ε	η	0
d_F^ω	0	0	1	1	0	0	0	0
d_F	-1	$d+1$	1	0	1	ε	η	0

canonical dimensions have been assigned to it), the definite canonical dimensions can be assigned to the other quantities. They are given in Table 1, including the dimensions of renormalized parameters, which will appear later on. From Table 1 it follows that the model is logarithmic (the coupling constants g_0 and u_0 are dimensionless) at $\varepsilon = \eta = 0$, and the UV divergences in the Green functions have the form of the poles in ε, η and their linear combinations.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma = \langle\Phi\cdots\Phi\rangle_{1-\mathrm{ir}}$ is given by the relation $d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences should be based on the following auxiliary considerations:

(i) All the 1-irreducible Green functions with $N_{\theta'} < N_\theta$ vanish; see Sec. 2. All the 1-irreducible functions with $N_{\theta'} = 0$ contain closed circuits of retarded propagators $\langle\theta\theta'\rangle_0$ and also vanish.

(ii) The field θ enters into the vertex $\theta'(\mathbf{v}\boldsymbol{\partial})\theta$ only in the form of the derivative, which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_\theta$ (the Green function requires counterterms only if d'_Γ is a non-negative integer) This also means that θ enters into the counterterms only in the form of the derivative $\partial\theta$.

From the dimensions in Table 1 we find $d_\Gamma = d + 2 - N_{\mathbf{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = d + 2 - N_{\mathbf{v}} - (d+1)N_{\theta'}$. Bearing in mind that $N_{\theta'} \geq N_\theta$ we conclude that for any d, superficial divergences can exist only in the 1-irreducible functions $\langle\theta'\rangle_{1-\mathrm{ir}}$ with $d_\Gamma = d'_\Gamma = 1$, $\langle\theta'\mathbf{v}\rangle_{1-\mathrm{ir}}$ with $d_\Gamma = d'_\Gamma = 0$, $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ with $d_\Gamma = 2$, $d'_\Gamma = 1$, and $\langle\theta'\theta\mathbf{v}\rangle_{1-\mathrm{ir}}$ with $d_\Gamma = 1$, $d'_\Gamma = 0$. The corresponding counterterms necessarily reduce to the forms $(\mathbf{h}\cdot\boldsymbol{\partial})\theta'$, $(\mathbf{h}\cdot\mathbf{v})\,\theta'$, $\theta'\partial^2\theta$ and $\theta'(\mathbf{v}\cdot\boldsymbol{\partial})\,\theta$, respectively. The first of these has the form of a total derivative, vanishes after the integration over $\mathbf{x}$ and therefore gives no contribution to the renormalized action:

$$S_R(\Phi) = \theta'\left[-\partial_t\theta + \nu Z_1\partial^2\theta - Z_2(\mathbf{v}\cdot\boldsymbol{\partial})\theta - Z_3(\mathbf{h}\cdot\mathbf{v})\right] - \mathbf{v}D_v^{-1}\mathbf{v}/2. \quad (3.1)$$

Here and below the dimensionless parameters g, u, and ν are the renormalized analogs of the bare parameters, μ is the renormalization mass in the minimal subtraction (MS) scheme, which we always use in practical calculations, and $Z_i = Z_i(g, u, \alpha)$ are the renormalization constants.

The original action (2.7) is invariant with respect to the transformations $\theta \to \theta + \mathbf{b}\cdot\mathbf{x}$, $\mathbf{h} \to \mathbf{h} - \mathbf{b}$ for any constant vector $\mathbf{b}$. This symmetry is preserved by the renormalization, so that the combination $(\mathbf{v}\cdot\boldsymbol{\partial})\,\theta + \mathbf{h}\cdot\mathbf{v}$ must enter into the renormalized action as a whole. This implies the exact relation $Z_2 = Z_3$.

The inclusion of the counterterms is reproduced by the multiplicative renormalization of the velocity field, $\mathbf{v} \to Z_2\mathbf{v}$, and the parameters g_0, u_0, and ν_0 in the action functional (2.7):

$$\nu_0 = \nu Z_\nu, \quad u_0 = u\mu^\eta\, Z_u \quad g_0 = g\mu^\varepsilon\, Z_g. \tag{3.2}$$

The constants in Eqs. (3.1) and (3.2) are related as follows:

$$Z_\nu = Z_1, \quad Z_u = Z_1^{-1}, \quad Z_g = Z_2^2 Z_1^{-2}. \tag{3.3}$$

The last two relations in Eq. (3.3) result from the absence of the renormalization of the term with D_v in (3.1). No renormalization of the fields θ, θ' and the parameters m, α, $\mathbf{h}$ is required, i.e., $Z_\theta = 1$ and so on.

In the following, we shall not be interested in the Green functions involving the velocity field $\mathbf{v}$, so that we can set $A^{\mathbf{v}} = 0$ in Eq. (2.9). Then the relation $S(\theta, \theta', Z_2\,\mathbf{v}, e_0) = S_R(\theta, \theta', \mathbf{v}, e, \mu)$ (where e_0 is the complete set of bare parameters, and e is the set of renormalized parameters) for the generating functional $W(A)$ in Eq. (2.8) yields $W(A, e_0) = W_R(A, e, \mu)$. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG equation:

$$\mathcal{D}_{RG}\, W_R(A, e, \mu) = 0, \tag{3.4}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_g\partial_g + \beta_u\partial_u - \gamma_\nu\mathcal{D}_\nu. \tag{3.5}$$

In Eq. (3.5), we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and the RG functions (the β functions and the anomalous dimensions γ) are defined as

$$\gamma_F \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F \tag{3.6}$$

for any renormalization constant Z_F and

$$\beta_g \equiv \widetilde{\mathcal{D}}_\mu g = g[-\varepsilon + 2\gamma_1 - 2\gamma_2], \quad \beta_u \equiv \widetilde{\mathcal{D}}_\mu u = u[-\eta + \gamma_1]. \tag{3.7}$$

The relations between β and γ result from the definitions and the relations (3.3).

Now let us turn to the explicit calculation of the constants $Z_{1,2}$ in the one-loop approximation. They are determined by the requirement that the 1-irreducible Green functions $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ and $\langle\theta'\theta\mathbf{v}\rangle_{1-\mathrm{ir}}$ be UV finite when expressed in renormalized variables (i.e., have no singularities for ε, $\eta \to 0$). The first of these functions satisfies the Dyson equation:

$$\langle\theta'\theta\rangle_{1-\mathrm{ir}} = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{\theta'\theta}(\omega, k), \tag{3.8}$$

$$\Sigma_{\theta'\theta} =$$

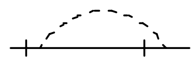


Figure 1. One-loop approximation of the self-energy operator in model (2.7).

where $\Sigma_{\theta'\theta}$ is the self-energy operator represented by the corresponding 1-irreducible diagrams. In the one-loop approximation is shown in Fig. 1. Here and below the solid lines in the diagrams denote the bare propagator $\langle\theta\theta'\rangle_0$ from Eq. (2.10), the end with a slash corresponds to the field θ', and the end without a slash corresponds to θ; the dashed lines denote the bare propagator (2.3); the vertices correspond to the factor V_i, see Sec. 2. The analytic expression for the diagram in Fig. 1 has the form

$$\Sigma_{\theta'\theta}(\omega,k) = -k_i\int\frac{d\omega'}{2\pi}\int\frac{d\mathbf{q}}{(2\pi)^d}\,(k_j+q_j)\,\frac{D_v(\omega',q)\,[P_{ij}(\mathbf{q})+\alpha Q_{ij}(\mathbf{q})]}{-\mathrm{i}(\omega+\omega')+\nu_0(\mathbf{q}+\mathbf{k})^2}\,, \tag{3.9}$$

where $q\equiv|\mathbf{q}|$ and $D(\omega',q)$ is given by Eq. (2.3), the factor $k_i(k_j+q_j)$ appears from the vertex factors V_i. We recall that the integration over q in Eq. (3.9) is restricted from below at $q\simeq m$. The integration over ω' yields

$$\Sigma_{\theta'\theta}(\omega,k) = -k_i\frac{g_0\nu_0^2}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\,(k_j+q_j)\frac{q^{2-d-\varepsilon}[P_{ij}+\alpha Q_{ij}]}{-\mathrm{i}\omega+\nu_0(\mathbf{q}+\mathbf{k})^2+u_0\nu_0q^{2-\eta}}\,. \tag{3.10}$$

Equation (3.10) gives the explicit expression for the self-energy operator in the first order $O(g_0)$ of the unrenormalized perturbation theory. Now we need to find $\Sigma_{\theta'\theta}$ in the order $O(g)$ of the renormalized perturbation theory; therefore we should simply replace $g_0\to g\mu^\varepsilon$, $u_0\to u\mu^\eta$, $\nu_0\to\nu$ in Eq. (3.10). In the bare term $\nu_0k^2=\nu Z_1k^2$ in Eq. (3.8) the $O(g)$ contribution to Z_1 should be retained. We know that the divergent part of the diagram is independent of ω, so that we set $\omega=0$ in what follows. Since the counterterm is proportional to k^2, we expand the integrand in Eq. (3.10) and neglect all the terms higher than k^2:

$$\Sigma_{\theta'\theta}(\omega=0,k) = -k_i\frac{g\mu^\varepsilon\nu}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\,(k_j+q_j)\frac{q^{2-d-\varepsilon}[P_{ij}+\alpha Q_{ij}]}{(\mathbf{q}+\mathbf{k})^2+uq^2(\mu/q)^\eta}\simeq$$

$$\simeq -k_ik_j\frac{g\mu^\varepsilon\nu}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\,\frac{q^{-d-\varepsilon}[P_{ij}+\alpha Q_{ij}]}{1+u(\mu/q)^\eta}\left\{1-\frac{2\alpha}{1+u(\mu/q)^\eta}\right\}, \tag{3.11}$$

where $\simeq$ denotes the equality up to an UV finite part, and only the contributions even in k_i are retained in the integrand. Using the relation

$$\int d\mathbf{q}\,f(q)\frac{q_iq_j}{q^2} = \frac{\delta_{ij}}{d}\int d\mathbf{q}\,f(q)$$

we obtain

$$\Sigma_{\theta'\theta}(\omega=0,k) = k^2\,\frac{g\mu^\varepsilon\nu}{2d}\Big[-(d-1+\alpha)J_0+2\alpha J_1\Big], \tag{3.12}$$

where we have written

$$J_0\equiv\int\frac{d\mathbf{q}}{(2\pi)^d}\,\frac{q^{-d-\varepsilon}}{1+u\,(\mu/q)^\eta},\quad J_1\equiv\int\frac{d\mathbf{q}}{(2\pi)^d}\,\frac{q^{-d-\varepsilon}}{[1+u\,(\mu/q)^\eta]^2}. \tag{3.13}$$

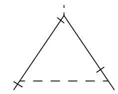
$$\langle \theta' \theta v_i \rangle_{1-\mathrm{ir}} = V_i\, Z_2 \; + $$ 

Figure 2. One-loop approximation of the function $\langle \theta' \theta \mathbf{v} \rangle_{1-\mathrm{ir}}$ in model (2.7).

The renormalized function $\langle \theta' \theta \mathbf{v} \rangle_{1-\mathrm{ir}}$ in the one-loop approximation is represented as shown in Fig. 2. Proceeding as above for $\langle \theta' \theta \rangle_{1-\mathrm{ir}}$ we obtain

$$\langle \theta' \theta v_i \rangle_{1-\mathrm{ir}} = V_i \left\{ Z_2 - \frac{g \mu^{\varepsilon} \alpha}{2d}\, J_1 \right\} \tag{3.14}$$

with J_1 from (3.13). The integrals over $\mathbf{q}$ in Eq. (3.13) are easily performed using the expansion in u:

$$\begin{aligned} J_0 &= \sum_{l=0}^{\infty} (-u)^l \mu^{l\eta} \int \frac{d\mathbf{q}}{(2\pi)^d} q^{-d-\varepsilon-l\eta} = C_d\, m^{-\varepsilon} \sum_{l=0}^{\infty} (\mu/m)^{l\eta}\, \frac{(-u)^l}{\varepsilon + l\eta}\,, \\ J_1 &= \sum_{l=0}^{\infty} (-u)^l \mu^{l\eta}\, (l+1) \int \frac{d\mathbf{q}}{(2\pi)^d} q^{-d-\varepsilon-l\eta} = \\ &= C_d\, m^{-\varepsilon} \sum_{l=0}^{\infty} (\mu/m)^{l\eta}\, \frac{(-u)^l\, (l+1)}{\varepsilon + l\eta}\,, \end{aligned} \tag{3.15}$$

where the parameter m arises from the IR limit in the integral over $\mathbf{q}$, $C_d \equiv S_d/(2\pi)^d$ and $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d-dimensional space.

The renormalization constants are found from the requirement that the UV divergences cancel out in expressions (3.8) and (3.14). This determines $Z_{1,2}$ up to UV finite contributions; the latter are fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form "1 + only poles in ε, η and their linear combinations," which gives the following expressions:

$$Z_1 = 1 - \frac{g\, C_d\, (d-1+\alpha)}{2d}\, \mathcal{S}_0 + \frac{g\, C_d\, \alpha}{d}\, \mathcal{S}_1, \quad Z_2 = 1 + \frac{g\, C_d\, \alpha}{2d}\, \mathcal{S}_1, \tag{3.16}$$

with C_d from Eq. (3.15) and

$$\mathcal{S}_0 = \sum_{l=0}^{\infty} \frac{(-u)^l}{\varepsilon + l\eta}, \quad \mathcal{S}_1 = \sum_{l=0}^{\infty} \frac{(-u)^l\, (l+1)}{\varepsilon + l\eta}. \tag{3.17}$$

The one-loop RG functions can be calculated from the renormalization constants (3.16) using the identity $\widetilde{\mathcal{D}}_{\mu} = \beta_g \partial_g + \beta_u \partial_u$, which follows from the definitions (3.6), (3.7) and the fact that $Z_{1,2}$ depend only on the charges g, u. Within our accuracy this identity reduces to $\widetilde{\mathcal{D}}_{\mu} \simeq -\varepsilon \mathcal{D}_g - \eta \mathcal{D}_u$. From Eq. (3.16) using the relations

$$\Big[\varepsilon \mathcal{D}_g + \eta \mathcal{D}_u\Big] g \mathcal{S}_0 = \sum_{l=0}^{\infty} (-u)^l = \frac{g}{1+u}, \quad \Big[\varepsilon \mathcal{D}_g + \eta \mathcal{D}_u\Big] g \mathcal{S}_1 = \sum_{l=0}^{\infty} (-u)^l (l+1) = \frac{g}{(1+u)^2}$$

one obtains:

$$\gamma_1 = \frac{g\, C_d}{2d(1+u)} \left[(d-1+\alpha) - \frac{2\alpha}{(1+u)} \right], \quad \gamma_2 = -\frac{g\, C_d\, \alpha}{2d(1+u)^2}, \tag{3.18}$$

up to corrections of order g^2 and higher. The beta functions are then obtained from Eqs. (3.18) using the relations (3.7).

4. Fixed points and scaling regimes

It is well known that possible scaling regimes of a renormalizable model are associated with the IR stable fixed points of the corresponding RG equations. The coordinates g_*, u_* of the fixed points are found from the equations

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0 \tag{4.1}$$

with the beta functions given in Eqs. (3.7). The type of a fixed point is determined by the matrix $\Omega = \{\Omega_{ij} = \partial\beta_i/\partial g_j\}$, where β_i denotes the full set of the beta functions and g_j is the full set of charges. For IR stable fixed points the matrix Ω is positive, i.e., the real parts of all its eigenvalues are positive.

The analysis of the beta functions (3.7) reveals five fixed points, which we denote FPI, FPII, and so on. In order to find the first two of them, it is convenient to introduce the new variables $w \equiv 1/u$, $g' \equiv g/u$; the corresponding beta functions have the form

$$\begin{aligned} \beta_w &\equiv \widetilde{\mathcal{D}}_\mu w = -\beta_u/u^2 = w[\eta - \gamma_1], \\ \beta_{g'} &\equiv \widetilde{\mathcal{D}}_\mu g' = \beta_g/u - g\beta_u/u^2 = g'[\eta - \varepsilon + \gamma_1 - 2\gamma_2], \end{aligned} \tag{4.2}$$

and the anomalous dimensions (3.18) are written as

$$\gamma_1 = \frac{g'\, C_d}{2d(1+w)} \left[(d-1+\alpha) - \frac{2\alpha w}{(1+w)} \right], \quad \gamma_2 = -\frac{g' w\, C_d\, \alpha}{2d(1+w)^2}. \tag{4.3}$$

From Eqs. (4.2) and (4.3) we find two fixed points. The first point is trivial,

$$\mathrm{FPI}: \qquad w_* = g'_* = 0; \quad \gamma^*_{1,2} = 0. \tag{4.4}$$

The corresponding matrix Ω is diagonal with the diagonal elements

$$\Omega_1 = \eta, \quad \Omega_2 = \eta - \varepsilon. \tag{4.5}$$

For the second point we obtain

$$\mathrm{FPII}: \qquad w_* = 0, \quad g'_* C_d = \frac{2d(\varepsilon - \eta)}{(d-1+\alpha)}; \quad \gamma^*_1 = \varepsilon - \eta, \quad \gamma^*_2 = 0. \tag{4.6}$$

The corresponding matrix Ω is triangular, $\partial_{g'}\beta_w = 0$, and its eigenvalues coincide with the diagonal elements:

$$\Omega_1 = \partial_w \beta_w = \eta - \gamma^*_1 = 2\eta - \varepsilon, \quad \Omega_2 = \partial_{g'}\beta_{g'} = \varepsilon - \eta. \tag{4.7}$$

Now we return to the original variables g, u. From Eqs. (3.7), (3.18) we find three more fixed points. The first one is trivial,

$$\mathrm{FPIII}: \qquad u_* = g_* = 0; \quad \gamma^*_{1,2} = 0. \tag{4.8}$$

The corresponding matrix Ω is diagonal with the elements

$$\Omega_1 = -\eta, \quad \Omega_2 = -\varepsilon. \tag{4.9}$$

For the point FPIV we obtain

$$\text{FPIV}: \qquad u_* = 0, \quad g_* C_d = \frac{\varepsilon d}{(d-1)};$$
$$\gamma_1^* = \gamma_2^* - \varepsilon/2 = \frac{\varepsilon(d-1-\alpha)}{2(d-1)}. \tag{4.10}$$

The corresponding matrix Ω is triangular, $\partial_{g'}\beta_u = 0$, and its eigenvalues have the form

$$\Omega_1 = \partial_u \beta_u = -\eta + \gamma_1^*, \quad \Omega_2 = \partial_g \beta_g = \varepsilon. \tag{4.11}$$

For the last fixed point FPV both the coordinates g_*, u_* are finite:

$$\text{FPV}: \qquad \frac{g_* C_d}{(1+u_*)} = \frac{2d(\varepsilon-\eta)}{(d-1+\alpha)}, \quad u_* = -1 + \frac{2\alpha(\eta-\varepsilon)}{(d-1+\alpha)(2\eta-\varepsilon)},$$
$$\gamma_1^* = \eta, \quad \gamma_2^* = \eta - \varepsilon/2. \tag{4.12}$$

A cumbersome but straightforward analysis shows that the positivity condition for Ω and the condition $u_* > 0$ (required by the physical meaning of u) are satisfied only in the region on the ε–η plane specified by the inequalities $\varepsilon > 0$, $\varepsilon(d-1-\alpha) < 2\eta(d-1)$.

It is noteworthy that some of the expressions given above are exact, i.e., they have no corrections of order ε^2, $\varepsilon\eta$, and so on, as a consequence of the exact relations between β and γ in (3.6) and (3.7). These are all the results for the trivial points, Ω_1 and $\gamma_{1,2}^*$ for FPII, the relations between Ω_1 and $\gamma_{1,2}^*$ for FPIV and $\gamma_{1,2}^*$ for FPV. The expression for γ_2^* in Eq. (4.10) is exact only for the incompressible case $\alpha = 0$.

It is clear from the Eqs. (2.5), (2.6) that the critical regime governed by the point FPII corresponds to the rapid-change limit (2.5) of our model, while the point FPIV corresponds to the limit of the frozen velocity field; see Eq. (2.6). Note that the expression for g'_* in Eq. (4.10) coincides with the exact expression obtained in [18] directly for the rapid-change model, and is therefore also exact. We then expect that all the critical dimensions at the point FPII [FPIV] depend on the only exponent $\zeta \equiv \varepsilon - \eta$ [ε] that survives in the limit in question, and coincide with the corresponding dimensions obtained directly for the models (2.5) [(2.6)]. This is indeed the case; see Eqs. (4.15) and (5.3) below. To avoid possible misunderstandings we emphasize that the limits g_0, $u_0 \to \infty$ or $u_0 \to 0$ are not supposed in the original correlation function (2.3); the parameters g_0, u_0 are fixed at some finite values. The behaviour characteristic of the models (2.5), (2.6) arises asymptotically in the regimes FPII, FPIV as a result of the solution of the RG equations, when the "RG flow" approaches the corresponding fixed point. More detailed discussion of the physical meaning of the fixed points can be found in [13] for the incompessible case. Triviality of the points FPI and FPIII implies the absence of anomalous scaling; we shall not dwell on these regimes in what follows.

In Fig. 3, we show the regions of stability for the fixed points FPI–FPV in the ε–η plane (i.e., the regions for which the eigenvalues of the Ω matrix are positive), for some value of $\alpha > 0$. The boundaries of the regions are depicted by thick lines. In the approximation linear in ε, η, the regions adjoin each other without overlaps or gaps. When α increases, the boundary between FPIV and FPV (i.e., the ray

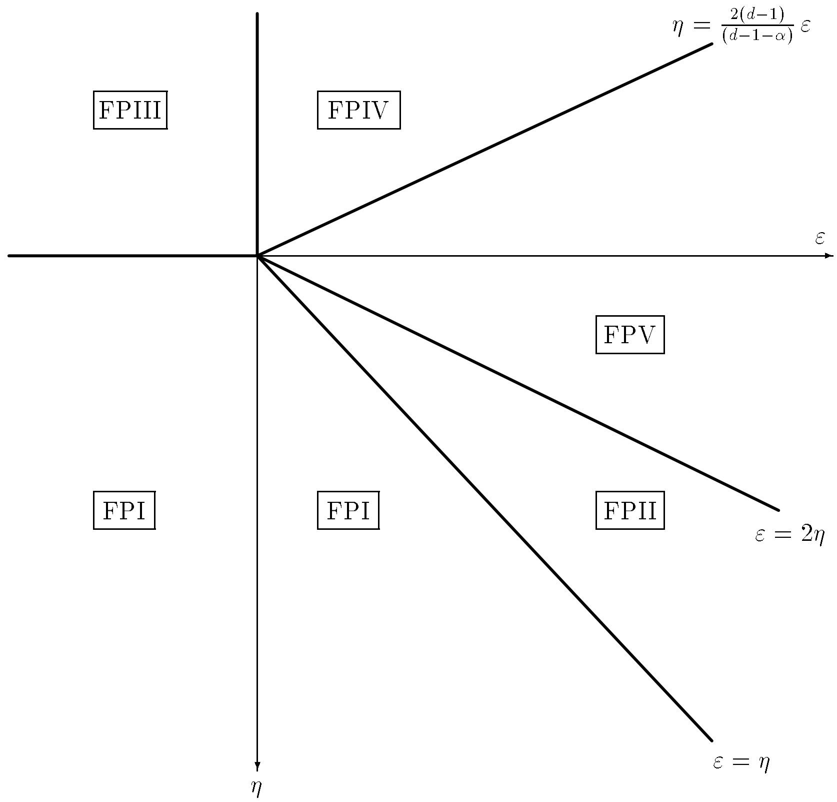


Figure 3. Regions of stability of the fixed points FPI–FPV in model (2.7).

$2\eta(d-1) = \varepsilon(d-1-\alpha)$, $\varepsilon > 0$) moves upwards and for $\alpha \to \infty$ coincides with the boundary between FPIII and FPIV ($\varepsilon = 0$, $\eta < 0$), so that the region of stability for FPIV shrinks to zero size. Moreover, the fixed point (4.10) for $\alpha \to \infty$ disappears. In order to treat this limit accurately, it is necessary to change to the new variable $g'' = g\alpha$ and keep it finite as $\alpha \to \infty$. Then the beta function $\beta_{g''} = \alpha\beta_{g''}$ for $\alpha \to \infty$, $u = 0$ becomes linear, $\beta_{g''} = -\varepsilon g''$, and has no fixed points other than $g'' = 0$. This fact holds valid for all orders of the perturbation theory, which becomes clear from the comparison with the models of the random walks in random environment with long-range correlations; see Refs. [27, 28, 29] and references therein.‡

When α decreases, this boundary moves clockwise and for $\alpha = 0$ coincides with the ray $\varepsilon = 2\eta > 0$, the boundary between FPII and FPIV. Hence, the region of stability for FPV turns into the ray $\varepsilon = 2\eta > 0$, in agreement with the result obtained in [13] directly for the incompressible case. In the latter, the coordinates of the fixed point FPV and the anomalous exponents are nonuniversal in the sense that they can depend not only on the exponent $\varepsilon = 2\eta$, but also on the amplitudes g_0, u_0; see [11, 13]. This nonuniversality can be viewed as a consequence of the ambiguity of the values of g_*, u_* in Eq. (4.12) in the limit $\alpha \to 0$, $\varepsilon \to 2\eta$. The limit of the combination $g_* C_d/(1+u_*) = 2d\eta/(d-1)$ is

‡ Strictly speaking, the model (2.1) in the frozen limit (2.6) differs from the model of random-random walks studied in [27, 28, 29], but their basic RG functions coincide; see Sec. 6.

unambiguous and coincides up to the notation with the result obtained in [13] directly for the incompressible case.

Let $F(r)$ be some equal-time two-point quantity, for example, the pair correlation function of the primary fields θ, θ' or some composite operators. We assume that $F(r)$ is multiplicatively renormalizable, i.e., $F = Z_F F^R$ with certain renormalization constant Z_F. The existence of nontrivial IR stable fixed point implies that in the IR asymptotic region $\Lambda r >> 1$ and any fixed mr the function $F(r)$ takes on the form

$$F(r) \simeq \nu_0^{d_F^\omega}\, \Lambda^{d_F} (\Lambda r)^{-\Delta_F}\, \xi(mr), \tag{4.13}$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F, respectively, and ξ is some function whose explicit form is not determined by the RG equation itself. The critical dimension Δ_F is given by the expression

$$\Delta[F] \equiv \Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \tag{4.14}$$

where γ_F^* is the value of the anomalous dimension (3.6) at the fixed point and $\Delta_\omega = 2 - \gamma_\nu^*$ is the critical dimension of frequency. From Eqs. (3.3) and (3.6) it follows $\gamma_\nu = \gamma_1$, so that for the nontrivial fixed points we obtain

$$\Delta_\omega = 2 - \begin{cases} \zeta & \text{for FPII and FPV,} \\ \dfrac{\varepsilon(d-1-\alpha)}{2(d-1)} + O(\varepsilon^2) & \text{for FPIV,} \end{cases} \tag{4.15}$$

(we recall that $\zeta \equiv \varepsilon - \eta$, see (2.5)). The results for FPII and FPV are exact and coincide with their analogs for the incompressible case.

Note that the dimensions d_F^ω, d_F and Δ_F of the pair correlator $F(r) = \langle F_1(x) F_2(x')\rangle$ are equal to the sums of the corresponding dimensions of the quantities $F_{1,2}$. The critical dimensions of the fields θ, θ' in our model are found exactly: $\Delta_\theta = -1 \quad \Delta_{\theta'} = d+1$, and for the IR scale we have $\Delta_m = 1$ (we recall that all these quantities in the model (2.7) are not renormalized and therefore their anomalous dimensions vanish identically, $\gamma_F = 0$).

The simplest composite operators $\theta^n(x)$ in the model are UV finite, $\theta^n = Z_n\, [\theta^n]^R$ with $Z_n = 1$. It then follows that the critical dimension of θ^n is simply given by the expression (4.14) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors: $\Delta[\theta^n] = n\Delta[\theta] = -n$. The proof is almost identical to the analogous proof for the Kraichnan model, given in [10], and will not be discussed here. Note also that these dimensions are independent of α and coincide with their analogs for the incompressible case.

5. Operator product expansion and the anomalous scaling

The representation (4.13) for any scaling function $\xi(mr)$ describes the behaviour of the Green function for $\Lambda r >> 1$ and any fixed value of mr. The inertial range corresponds to the additional condition that $mr << 1$. The form of the function $\xi(mr)$ is not determined by the RG equations themselves; its behaviour for $mr \to 0$ is studied using the well-known Wilson operator product expansion, see, e.g., [24, 25], and has the form:

$$\xi(mr) = \sum_F A_F \, (mr)^{\Delta_F}, \tag{5.1}$$

where the coefficients $A_F = A_F(mr)$ are regular in $(mr)^2$. The summation in Eq. (5.1) runs over all possible composite operators $F(x)$ which contribute to the OPE, Δ_F being their critical dimensions.

In the following, an important role will be played by the critical dimensions $\Delta[n,p]$, associated with the tensor composite operators of the form

$$F[n,p] \equiv \partial_{i_1}\theta \cdots \partial_{i_p}\theta \, (\partial_i\theta\partial_i\theta)^l, \tag{5.2}$$

where p is the number of the free vector indices and $n = p + 2l$ is the total number of the fields θ entering into the operator; the vector indices of the symbol $F[n,p]$ are omitted. The analysis similar to that given in Ref. [13] shows that these operators mix only with each other in renormalization, the corresponding renormalization matrices are triangular, and the critical dimensions associated with the family (5.2) are given by $\Delta[n,p] = \gamma^*[n,p]$, where $\gamma[n,p] = \widetilde{\mathcal{D}}_\mu \ln Z[n,p]$ and $Z[n,p]$ is the diagonal element of the renormalization matrix, corresponding to the operator $F[n,p]$. The "basis" operator that possesses definite critical dimension $\Delta[n,p]$ is a p-th rank tensor (traceless for $p \geq 2$), given by a linear combination of the monomials $F[n',p']$ with $n' \leq n$, $p' \leq p$; the coefficients involve the vector $\mathbf{h}$ and Kronecker delta symbols.

The one-loop calculation of the dimensions $\Delta[n,p]$ is also similar to that performed in Ref. [13], and below we present only the result. For the nontrivial fixed points discussed in Sec. 3 we have obtained

$$\Delta[n,p] = \frac{Q[n,p]}{4(d+2)} \times \begin{cases} 2\zeta/(d-1+\alpha) & \text{for FPII and FPV,} \\ \varepsilon/(d-1) & \text{for FPIV,} \end{cases} \tag{5.3}$$

up to corrections of order ε^2 and so on. In Eq. (5.3), we have denoted

$$Q[n,p] = 2n(n-1)(1-\alpha) - (n-p)(n+p+d-2)(d+1+\alpha). \tag{5.4}$$

The expression (5.3) illustrates the general fact that the critical dimensions in the rapid-change and frozen regimes depend only on the exponents ζ and ε, respectively. The coincidence of the results for the points FPIV and FPV seems to be an artifact of the one-loop approximation. For $\alpha = 0$, the results of Ref. [13] are recovered. We also note that the dimensions (5.3) have well-defined limit $d \to 1$ for $p = 0$ and 1, i.e., for the scalar and vector operators, and this limit does not depend on α.

The result $\Delta[2,0] = -\zeta$ for the rapid-change regime is in fact exact. The proof is based on certain Schwinger equation; it resembles the analogous proof for the Kraichnan model with $\alpha = 0$, given in Ref. [10], and will not be discussed here.

Now let us turn to the equal-time structure functions, which are defined by the relations

$$S_n(r) \equiv \langle [\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^n \rangle, \quad r \equiv |\mathbf{x} - \mathbf{x}'|. \tag{5.5}$$

For these, the representation (4.13) is valid with the dimensions $d_F^\omega = 0$ and $d_F = \Delta_F = n\Delta_\theta = -n$.

In general, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. The leading term of the Taylor expansion for the function (5.5) is given by the n-th rank tensor $F[n,n]$ from Eq. (5.2). The decomposition of $F[n,n]$ in irreducible tensors gives rise to the operators $F[n,p]$ with all possible values of $p \le n$; the admixture of junior operators gives rise to all the monomials $F[k,p]$ with $k < n$ and all possible p. Hence, the desired asymptotic expression for the structure function has the form

$$S_n(r) \simeq (hr)^n \sum_{k=0}^{n} \sum_{p=p_k}^{k} \Big[C_{kp}\,(mr)^{\Delta[k,p]} + \cdots \Big], \tag{5.6}$$

with the dimensions $\Delta[k,p]$ from Eq. (5.3). Here and below p_k denotes the minimal possible value of p for given k, i.e., $p_k = 0$ for k even and $p_k = 1$ for k odd; C_{kp} are some numerical coefficients dependent on ε, η, d, α, and on the angle ϑ between the vectors $\mathbf{h}$ and $\mathbf{r}$. The dots in Eq. (5.6) stand for the contributions of order $(mr)^{2+O(\varepsilon)}$ and higher, which arise from the senior operators like $\partial^2\theta\partial^2\theta$. The operators $F[k,p]$ with $k > n$ (whose contributions would be more important) do not appear in Eq. (5.6), because they are absent in the Taylor expansion of S_n and do not admix in renormalization to the terms of the Taylor expansion.

The mean value of the operator with the dimension $\Delta[k,p]$ is a traceless (for $p \ge 2$) tensor built of the vector $\mathbf{h}$ and Kronecker delta symbols; the contraction with the vector indices of the corresponding coefficient $C_\alpha(\mathbf{r})$ in Eq. (5.1) gives rise to p-th order Legendre polynomial $P_p(\cos\vartheta)$. The expansion (5.6) is thus consistent with the decomposition in the Legendre polynomials, or, more general, with the decomposition in the irreducible representations of the rotation group, employed, e.g., in Refs. [23].

The straightforward analysis of the explicit one-loop expression (5.3) shows that for fixed n, any $\alpha > 0$, and any nontrivial fixed point, the dimension $\Delta[n,p]$ decreases monotonically with p and reaches its minimum for the minimal possible value of $p = p_n$, i.e., $p = 0$ if n is even and $p = 1$ if n is odd:

$$\Delta[n,p] > \Delta[n,p'] \quad \text{if} \quad p > p'. \tag{5.7}$$

Similar inequalities are satisfied by the critical dimensions of certain tensor operators in the stirred Navier–Stokes turbulence, see Ref. [30] and Sec. 2.3 of [25], and in the model describing passive advection of the magnetic field by the rapid-change velocity in the presence of a constant background field [20]. Furthermore, this minimal value $\Delta[n,p_n]$ is negative§ and decreases monotonically as n increases:

$$0 > \Delta[2k,0] > \Delta[2k+1,1] > \Delta[2k+2,0]. \tag{5.8}$$

§ The existence of operators (5.2) with negative critical dimensions does not violate the IR stability of the fixed points, discussed in Sec. 4, because such operators do not appear in the renormalized action as counterterms: the latter must involve at least one field θ' and no more than one field θ (see Sec. 3).

The inequalities (5.7), (5.8) show that the contributions of the tensor operators (5.2) into the asymptotic expression (5.6) exhibit a kind of hierarchy: the less is the rank, the more important is the contribution; cf. [13] for the purely incompressible case.

The leading term of the expression (5.6) for the even (odd) function S_n is determined by the scalar (vector) composite operator consisting of n factors $\partial\theta$ and has the form

$$S_n(r) \propto (hr)^n\,(mr)^{\Delta[n,p_n]}. \tag{5.9}$$

It is easily seen from the explicit expression (5.3) that Eq. (5.9) for the rapid-change regime and even n is in agreement with the results obtained in Refs. [16, 17] for a "tracer" and earlier in [19] for $d = 1$.‖

Expressions analogous to Eqs. (5.6), (5.9) can be written down for other correlation functions, provided their canonical and critical dimensions are known; in particular, the analog of the expression (5.9) for the equal-time pair correlator of the operators (5.2) has the form

$$\langle F[n,p]\,F[n',p']\rangle \simeq h^{n+n'}\,(\Lambda r)^{-\Delta[n,p_n]-\Delta[n',p_{n'}]}(mr)^{\Delta[n+n',\,p_{n+n'}]}\,, \tag{5.10}$$

with the dimensions $\Delta[n,p]$ from Eq. (5.3); cf. [13] for the incompressible case.

6. Passive advection of a density field

So far, we have studied the model (2.1) with the nonlinearity of the form $(\mathbf{v}\cdot\boldsymbol{\partial})\,\theta$, which describes a "tracer" in the terminology of Refs. [16, 17]. Another possibility is to choose $\boldsymbol{\partial}(\mathbf{v}\theta)$, so that the dynamical equation

$$\partial_t\theta + \boldsymbol{\partial}(\mathbf{v}\theta) = \nu_0\partial^2\theta - \partial_i(v_i\,(\mathbf{h}\cdot\mathbf{v})) \tag{6.1}$$

for $\mathbf{h} = 0$ becomes a conservation law for θ; see [16, 18]. The "frozen" limit (2.6) of Eq. (6.1) with $\mathbf{h} = 0$ leads to the model of the random walks in random environment with long-range correlations; see [27, 28, 29].

For $\mathbf{h} = 0$, the action functionals corresponding to the models (2.1) and (6.1) are connected by the transformations $\theta \leftrightarrow \theta'$ and $t \to -t$. Since the field $\mathbf{h}$ does not enter into the diagrams needed for the calculation of the constants $Z_{1,2}$ (see Sec. 3), the basic RG functions, dimensions $\Delta_{\omega,\theta,\theta'}$, coordinates of the fixed points and their regions of stability for the models (2.1) and (6.1) coincide identically. Note that the expression (4.15) for FPIV is in agreement with the result obtained in the model of random-random walks [27, 28, 29] (up to notation and a misprint in Eq. (4.48) of Ref. [28]: 2ν in the first line should be replaced with $(2\nu)^{-1}$; in our notation, $\Delta_\omega = 1/\nu$).

The symmetry $\theta \to \theta + \mathrm{const}$, specific to the model (2.1), ceases to hold for the model (6.1), and the real index of divergence for this case has the form $d'_\Gamma = d_\Gamma - N_{\theta'}$ (the derivative ∂ at the vertex $\theta'\boldsymbol{\partial}(\mathbf{v}\theta)$ can be moved onto the field θ' using the integration by parts). As a result, the analysis of composite operators differs essentially in the two cases: in particular, the operators θ^n in the model (6.1) require nontrivial renormalization, $\theta^n = Z_n\,[\theta^n]^R$ with $Z_n \neq 1$; cf. [18] for the rapid-change case. Hence,

‖ In the notation of Refs. [16, 17], $\wp = \alpha/(d-1+\alpha)$.

the operators θ^n acquire nontrivial anomalous dimensions $\gamma_n = \widetilde{\mathcal{D}}_\mu \ln Z_n$ (they should not be confused with $\gamma_{1,2}$ from Secs. 3, 4), and for the corresponding critical dimensions $\Delta_n = -n + \gamma_n^*$ the one-loop calculation gives:

$$\Delta_n = -n - \frac{\alpha n(n-1)\, d}{4} \times \begin{cases} 2\zeta/(d-1+\alpha) & \text{for FPII and FPV,} \\ \varepsilon/(d-1) & \text{for FPIV,} \end{cases} \tag{6.2}$$

cf. Eq. (5.3). The ζ^2 contribution to γ_n^* and the exact expression for γ_2^* in the rapid-change regime are given in Ref. [18].

Consider now the equal-time pair correlator $\langle \theta^p(t,\mathbf{x})\theta^k(t,\mathbf{x}')\rangle$. Substituting the relations $d_F^\omega = 0$ and $d_F = -(p+k)$ into the general expression (4.13) gives $\langle \theta^p\theta^k\rangle = \Lambda^{-(p+k)}(\Lambda r)^{\Delta_p+\Delta_k}\xi_{p,k}(mr)$, with Δ_n from Eq. (6.2) (here and below, we do not display the obvious dependence on h). The small mr behaviour of the scaling functions $\xi_{p,k}$ is found from Eq. (5.1). In contrast to the examples from Sec. 5, the composite operators in the expansion (5.1) can involve the field θ *without derivatives*. The leading term in Eq. (5.1) is then determined by the monomial θ^{p+k}, which gives

$$\langle \theta^p\theta^k\rangle \simeq \Lambda^{-(p+k)}(\Lambda r)^{-\Delta_p-\Delta_k}(mr)^{\Delta_{p+k}}. \tag{6.3}$$

Note that Eq. (6.3) contains nontrivial dependence on both the IR and UV scales.

Now let us turn to the structure functions (5.5) in the inertial range $\Lambda r >> 1$, $mr << 1$. From the expression (6.3) it follows

$$S_n(r) \simeq \Lambda^{-n}(m/\Lambda)^{\Delta_n}\left\{1 + \sum_{\substack{k+p=n \\ k,p\neq 0}} C_{kp}(\Lambda r)^{\Delta_n-\Delta_k-\Delta_p}\right\}, \tag{6.4}$$

where the coefficients C_{kp} are independent of the scales Λ, m and the separation r. It is obvious from the inequality $\Delta_k + \Delta_p > \Delta_{k+p}$, satisfied by the dimensions (6.2), that all the contributions in the sum in Eq. (6.4) vanish in the limit $\Lambda r \to \infty$, so that the leading term of the structure function does not depend on r and is simply given by $S_n \simeq \langle \theta^n \rangle$.

The situation changes for the purely solenoidal velocity field, $\alpha = 0$ in Eq. (2.2). In this case, the models (2.1) and (6.1) become identical, the dimensions (6.2) become linear in n, $\Delta_n = -n$, and the expression (6.3) becomes independent of r. The operators θ^n cannot appear in the OPE for the structure functions. This means that the contributions of the operators θ^n to the pair correlators (6.3) cancel out in the functions (5.5), and the IR behaviour of the latter is dominated by the operators constructed solely of the scalar gradients. The cancellation becomes possible owing to the fact that all the terms in the curly brackets in Eq. (6.4) become independent of Λr. In this case, the anomalous scaling of S_n is determined by the critical dimensions of the operators (5.2).

7. Conclusion

We have applied the field theoretic renormalization group and operator product expansion to the model of a passive scalar advected by a Gaussian self-similar nonsolenoidal velocity field with finite correlation time and in the presence of large-scale anisotropy, induced by the linear mean gradient. The energy spectrum of the velocity in the inertial range was taken in the form $E(k) \propto g_0 k^{1-\varepsilon}$, and the frequency at the wavenumber k scales as $u_0 k^{2-\eta}$. We have shown that, depending on the values of the exponents ε and η, the model exhibits three types of the inertial-range scaling regimes with nontrivial anomalous exponents.

The explicit asymptotic expressions for the structure functions and other correlation functions have been obtained; the anomalous exponents, determined by the critical dimensions of certain composite operators, have been calculated to the first order in ε and η in any space dimension. For the first scaling regime the exponents are the same as in the rapid-change limit of the model; for the second they are the same as in the model with time-independent (frozen) velocity field. For all these regimes, the anomalous exponents are nonuniversal through their dependence on α, the degree of compressibility. However, they are independent of the amplitudes g_0, u_0, in contradistinction with the case of a local turnover exponent ($\varepsilon = 2\eta$) in an incompressible flow [11, 13].

The most interesting result of the RG analysis is that the anomalous exponents, related to the anisotropic contributions to the inertial-range behaviour (or, in other words, critical dimensions of the tensor composite operators in the corresponding OPE) exhibit a kind of hierarchy related to the degree of anisotropy: the less is the rank, the less is the dimension and, consequently, the more important is the contribution; cf. [13] for the purely incompressible case. The leading term for the even (odd) correlation functions is then determined by the scalar (vector) composite operators (i.e., those having minimal possible rank). A similar hierarchy is demonstrated by the critical dimensions of certain tensor operators in the stirred Navier–Stokes turbulence, see Ref. [30] and Sec. 2.3 of [25], and in the model of the magnetic field advected passively by the rapid-change velocity [20]. The picture outlined above and in Refs. [13, 20, 21] for passively advected fields (a superposition of power laws with universal exponents and nonuniversal amplitudes) seems rather general, being compatible with that established recently in the field of Navier-Stokes turbulence, on the basis of numerical simulations of channel flows and experiments in the atmospheric surface layer, see Refs. [23] and references therein. It can be viewed as an analytic argument in favour of the Kolmogorov hypothesis on the restored local (small-scale) isotropy; see the discussion in Ref. [20].

Nevertheless, the anisotropy survives in the inertial range and reveals itself in *odd* correlation functions, in disagreement with what was expected on the basis of the cascade ideas. Consider the odd-order dimensionless ratios

$$\mathcal{R}_{2n+1} \equiv S_{2n+1}/S_2^{n+1/2} \propto (mr)^{\Delta[2n+1,1]-(n+1/2)\Delta[2,0]}, \tag{7.1}$$

the last expression following from Eq. (5.9). For the incompressible case, substituting the explicit one-loop expression (5.3) gives

$$\mathcal{R}_{2n+1} \propto (mr)^{\zeta\,(d+2-4n^2)/2(d+2)} \tag{7.2}$$

(for definiteness, we present the result only for the rapid-change regime; the dependence on n in the one-loop approximation is the same for all regimes). For $n = 1$ (skewness factor), the well-known result of Ref. [9] is recovered.

According to the classical Kolmogorov theory, the quantities $\mathcal{R}_{2n+1}$ are expected to decrease for $mr \to 0$. From Eq. (7.2) it follows that $\mathcal{R}_3$ decreases but slower

than expected on the basis of the cascade picture, while the higher-order ratios diverge as $mr \to 0$. The latter fact agrees with the recent findings of Ref. [22], where the passive advection by the two-dimensional Navier–Stokes velocity field was studied in a numerical experiment.

For general α, expressions (7.2) take on the form

$$\mathcal{R}_{2n+1} \propto (mr)^{\zeta\,[(d-1+\alpha)(d+2-4n^2)-8\alpha\,n^2]/2(d+2)(d-1+\alpha)}. \tag{7.3}$$

From (7.3) it follows that already $\mathcal{R}_3$ becomes divergent for $mr \to 0$ provided the compressibility is strong enough (namely, if $\alpha > (d-1)(d-2)/(10-d) + O(\zeta)$), while the divergence of the higher-order ratios becomes even faster as α increases.

The hierarchy expressed by the relation (5.7) (which can be rewritten as $\partial\Delta[n,p]/\partial p > 0$) also becomes less pronounced as α grows, $\partial^2\Delta[n,p]/\partial p\partial\alpha < 0$, and the anisotropic corrections in Eq. (5.6) become closer to each other and to the leading term.

The enhancing of the small-scale anisotropy also seems rather general, being also observed in a model of the magnetic field advected passively by the non-solenoidal velocity field, see Ref. [31].

The explicit inertial-range expressions (5.6), (5.10), (6.4) show, however, that only the amplitudes in power laws are affected by the anisotropy (or by the statistics of the stirring force in the original Kraichnan model, or by the initial and boundary conditions in a finite-size problem). The exponents are independent of $\mathbf{h}$ and can be calculated directly in the homogeneous model with $\mathbf{h} = 0$. It follows from the derivation of Eqs. (5.6), (5.10), (6.4), that this statement holds valid not only for the leading exponents, related to the dimensions of scalar operators (5.2) with $p = 0$, but also for all the anisotropic corrections, related to the tensor operators with $p \neq 0$! However, the anomalous exponents of the passive scalar become nonuniversal and acquire the dependence on the anisotropy parameters if the velocity field is taken to be anisotropic at small scales.

References

[1] A. S. Monin and A. M. Yaglom, Statistical Fluid Mechanics, Vol.2 (MIT Press, Cambridge, MA, 1975).

[2] U. Frisch, Turbulence: The Legacy of A. N. Kolmogorov (Cambridge University Press, Cambridge, 1995).

[3] K. R. Sreenivasan, Proc. Roy. Soc. London A 434 (1991) 165; K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid. Mech. **29** (1997) 435.

[4] M. Holzer and E. D. Siggia, Phys. Fluids **6** (1994) 1820; A. Pumir, Phys. Fluids **6** (1994) 2118; C. Tong and Z. Warhaft, Phys. Fluids **6** (1994) 2165.

[5] R. H. Kraichnan, Phys. Fluids **11** (1968) 945; J. Fluid. Mech. **64** (1974) 737; Phys. Rev. Lett. **72** (1994) 1016; **78** (1997) 4922.

[6] M. Chertkov, G. Falkovich, I. Kolokolov and V. Lebedev, Phys. Rev. E **52** (1995) 4924; M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76** (1996) 2706.

[7] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75** (1995) 3834; D. Bernard, K. Gawędzki and A. Kupiainen, Phys. Rev. E **54** (1996) 2564.

[8] B. I. Shraiman and E. D. Siggia, Phys. Rev. Lett. **77** (1996) 2463; A. Pumir, B. I. Shraiman and E. D. Siggia, Phys. Rev. E **55** (1997) R1263

[9] A. Pumir, Europhys. Lett. **34** (1996) 25; Europhys. Lett. **37** (1997) 529; Phys. Rev. E **57** (1998) 2914.

[10] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev, Phys. Rev. E **58** (1998) 1823.

[11] M. Chertkov, G. Falkovich and V. Lebedev, Phys. Rev. Lett. **76** (1996) 3707.

[12] G. Eyink, Phys. Rev. E **54** (1996) 1497.

[13] N. V. Antonov, Phys. Rev. E **60** (1999) 6691.

[14] T. Elperin, N. Kleeorin and I. Rogachevskii, Phys. Rev E **52** (1995) 2617; E **55** (1997) 2713; Phys. Rev. Lett. **76** (1996) 224.

[15] M. Vergassola and M. Avellaneda, Physica D **106** (1997) 148.

[16] K. Gawędzki and M. Vergassola, Physica D **138** (2000) 63.

[17] A. Celani, A. Lanotte and A. Mazzino, Phys. Rev. E **60** (1999) R1138.

[18] L. Ts. Adzhemyan and N. V. Antonov, Phys. Rev. E **58** (1998) 7381.

[19] M. Vergassola and A. Mazzino, Phys. Rev. Lett. **79** (1997) 1849.

[20] A. Lanotte and A.Mazzino, Phys. Rev. E **60** (1999) R3483.

[21] N. V. Antonov, A. Lanotte and A. Mazzino, Phys. Rev. E **61** (2000) 6586

[22] A. Celani, A. Lanotte, A. Mazzino and M. Vergassola, Phys. Rev. Lett. **84** (2000) 2385.

[23] I. Arad, B. Dhruva, S. Kurien, V. S. L'vov, I. Procaccia and K. R. Sreenivasan, Phys. Rev. Lett. **81** (1998) 5330; I. Arad, L. Biferale, I. Mazzitelli and I. Procaccia, Phys. Rev. Lett. **82** (1999) 5040; S. Kurien, V. S. L'vov, I. Procaccia and K.R. Sreenivasan, Phys. Rev. E**61** (2000) 407

[24] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev, Usp. Fiz. Nauk **166** (1996) 1257 [in Russian; English translation: Phys. Usp. **39** (1996) 1193].

[25] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasiliev, The Field Theoretic Renormalization Group in Fully Developed Turbulence (Gordon & Breach, London, 1999).

[26] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon, Oxford, 1989).

[27] J. P. Bouchaud, A. Comtet, A. Georges and P. Le Doussal, J. Phys. (Paris) **48** (1987) 1445; **49** (1988) 369.

[28] J. P. Bouchaud and A. Georges, Phys. Rep. **195** (1990) 127.

[29] J. Honkonen and E. Karjalainen, J. Phys. A: Math. Gen. **21** (1988) 4217; J. Honkonen, Yu. M. Pis'mak and A. N. Vasil'ev, J. Phys. A: Math. Gen. **21** (1989) L835; J. Honkonen and Yu. M. Pis'mak, J. Phys. A: Math. Gen. **22** (1989) L899.

[30] L. Ts. Adzhemyan, S. V. Borisenok and V. I. Girina, Theor. Math. Phys. **105** (1995) 1556.

[31] N. V. Antonov, J. Honkonen, A. Mazzino and P. Muratore-Ginanneschi, Phys. Rev. E**62** (2000) R5891