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INTRODUCTION TO CONTINUOUS GROUPS

Учебно-методическое пособие

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В учебно-методическом пособии рассмотрены основные понятия теории групп. Основное внимание уделено линейным группам Ли и алгебрам Ли. Даны основные определения теории представлений групп Ли. Изложение иллюстрируется различными примерами. Пособие предназначено для студентов-физиков 3-4-го курсов, обучающихся по теоретическим специализациям.

Contents

1 Introduction

There is a fascinating interplay between mathematics and physics. In some cases the need to understand and formulate a physical problem provides a stimulus for the development of the relevant mathematics, as for example in Newton's development of the calculus as a tool for calculating planetary orbits. Or another example of development of grassmanian analysis as a tool for supersymmetry in physics of elementary particles. In other cases a mathematical formalism already developed turns out to be tailor-made for physics. Thus the theory of vector spaces is exactly what one needs for the general formulation of quantum mechanics, and the representation theory of groups is precisely the mathematical framework needed when one considers the action of symmetry transformations on quantum systems.

There are many physical systems whose underlying dynamics has some symmetry. A good example is provided by the water molecule. There is clearly a symmetry between the two hydrogen ions, which may be interchanged without affecting the energy of the system. Again, there is a translation symmetry: the interaction between any two ions situated at $\vec{r}_1$ and $\vec{r}_2$ depends only on their relative separation $\vec{r}_1 - \vec{r}_2$ and not on their absolute positions. That is, the potential energy $V(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ is actually a function of $\vec{r}_1 - \vec{r}_2$ and $\vec{r}_1 - \vec{r}_3$ only, and the same is true of the kinetic energy, once the centre-of-mass energy has been subtracted off. Furthermore, the system has a rotational invariance whereby its energy is independent of its absolute orientaton: $V = V(|\vec{r}_1 - \vec{r}_2|, |\vec{r}_1 - \vec{r}_3|)$. Thus the underlying Hamiltonian, the classical expression for the energy, which becomes an operator in quantum mechanics, is invariant under a set of transformations of the coordinates, which includes reflections, translations and rotations.

These different types of transformation all have the common property that they form a group. The formal definition of a group will be given in the next Section, but the essential property is that the successive application of two such transformations gives another one, which we call the product (or the group product) there is an identity transformation, which is simply to do nothing, and each transformation has an inverse, which undoes the operation, i.e. the product of a transformation and its inverse is the identity.

The underlying symmetry of the dynamics may or may not be explicitly manifest in a given physical realization. In the realm of classical mechanics it is more often than not the case that a particular dynamical system does not exhibit the full symmetry. A good example of this is the orbit of a planet under gravity. Any

given particular planetary orbit is an ellipse. It is not even circular, which worried the ancients greatly, but perhaps they should have been more worried that even a circle does not respect the perfect spherical symmetry of the underlying Hamiltonian. The explanation is that the symmetry has somehow been broken by the initial conditions of the motion, which picked out first of all a plane in which the motion would take place, and then in that plane a direction, say that of the semi-major axis. Paradoxically, the facts that the motion remains fixed are due to conservation laws (of angular momentum and of the Runge-Lentz vector) which are consequences of the underlying symmetries.

It turns out in fact that the most important applications of group theory in physics are found not in classical mechanics but rather in quantum mechanics. There the ground state of a system usually does exhibit the full symmetry of the Hamiltonian, though a very important and interesting exception to this occurs in the phenomenon of spontaneous symmetry breaking, where, again, because of some uncontrollable perturbation of the initial conditions, one asymmetric solution is picked out of an infinite set of possible ones. Thus the fundamental interactions of the spins in a ferromagnet are rotationally symmetric, but when one is formed they align themselves in some particular direction. But as an example of the more usual scenario, consider the ground state of the hydrogen atom, the quantum mechanical equivalent of the planetary orbit problem. There, as you know from the elementary quantum mechanics, the ground state wave function gives a spherically symmetric probability distribution which indeed respects the spherical symmetry of $1/r$ potential.

As far as the excited states are concerned, the rotational symmetry of the problem means that they can be classified by the total angular momentum quantum number l and the magnetic quantum number m, which refers to the eigenvalue of its z component. Moreover, the energy does not depend on m (nor on l, but this is a special feature of $1/r$ potential). This makes perfect sense physically, since there is no preferred direction: the choice of the z axis was completely arbitrary. As far as mathematics is concerned, it means that we have a degenerate space of eigenfunctions ψ_{nlm}, with m ranging from $+j$ to $-l$, which all have the same energy and can be transformed into each other by rotation. We can take arbitrary linear combinations of the $2l+1$ eigenfunctions which are still eigenfunctions with the same energy and total angular momentum. They therefore form what is usually known as a vector space. The group of rotations in ordinary three-dimensional space induces transformations within this vector space, giving what is known as a representation, which can be realized by matrices, in this of dimension $2l+1 \times 2l+1$.

This consideration of symmetry transformations leads us naturally to the study of groups, many properties of which can be established generally, without reference to the particular nature of the group elements. When the transformations act on a quantum mechanical system, the appropriate mathematical framework is that of representation theory. This theory, when developed, will enable us to classify energy levels, their degeneracy, and how this is changed when the symmetry is reduced, to obtain selection rules which tell us when certain matrix elements are zero.

2 Main notions

2.1 Definition of group

First of all, we have to give the definition for the main objects of our lectures - groups, elements of groups, their properties etc. From the mathematical point of view a group is an abstract object with very precise meaning. This is a set of some elements such that four axioms must be fulfilled. These very general axioms give the basis for the rich theory which has a lot of applications in different branches of natural sciences. The theory does not depend actually on the nature of elements, but the most frequently occurring case corresponds to the sets of transformations in some specific space.

Definition.

By definition, a group G is a set of elements $\gamma \in G$, for which four "axioms of group" are satisfied:

1) For every pair of elements γ, γ' from G another element γ'', which is associated with γ, γ', exists in G. This operation $(\gamma, \gamma') \to \gamma''$ is usually called **multiplication** in the group, it is written as $\gamma'' = \gamma\gamma'$. The element γ'' is the product of γ with γ'.

2) For arbitrary triplet of elements of G - $\gamma, \gamma', \gamma''$ - the following relation - the associative law - is fulfilled:

$$(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma''). \tag{1}$$

3) Group G contains the identity element E such that:

$$\gamma E = E\gamma = \gamma$$

for arbitrary element $\gamma \in G$.

4) For each element $\gamma \in G$ an **inverse element** γ^{-1} exists such that

$$\gamma\gamma^{-1} = \gamma^{-1}\gamma = E$$

Remark. It is known [19] that one can formulate the properties 3) and 4) above for left identity and left inverse element only.

Examples.

1) *The group of multiplication of real numbers.* One can choose all real numbers (without zero) as the elements of group G. Multiplication group operation can be chosen as an ordinary multiplication. All four axioms are obviously satisfied. The number 1 plays the role of group identity, and the number t^{-1} plays the role of inverse element for the element t.

2) *The group of addition of real numbers.* One can take the same set of elements (but this time, including zero). Choosing ordinary addition as a group multiplication, one obtains different group. Now the group identity coincides with 0, and inverse element for c is $-c$.

3) *A specific finite matrix group.* Let G be a group of eight elements:

$$\pm I, \pm\sigma_1, \pm i\sigma_2, \pm\sigma_3,$$

where I is 2×2 unity matrix, and Pauli matrices are defined as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{2}$$

Let us take a matrix multiplication as a group operation. It is easy to check that I can be considered as a unity element in the group G, while all eight elements coincide with their inverse.

Let us note that the matrix multiplications always obey the associative axiom 2) above. The example, when associative rule does not work, is the vector product of 3-dimensional vectors.

The most general set of matrices includes all $N \times N$ (real or complex) matrices A with the only condition $detA \neq 0$. All group axioms are obviously satisfied for these sets. Several particular matrix groups should be listed here.

1) The general linear matrix group $GL(n, C)$ of matrices $n \times n$ with complex elements. It depends on $2n^2$ real parameters. Analogous group $GL(n, R)$ of matrices with real elements depends on n^2 real parameters.

2) The special linear groups $SL(n, C)$ and $SL(n, R)$, which include matrices with additional restriction $detA = 1$, sometimes is called as special unimodular group.

3) The unitary group $U(n)$ of $n \times n$ complex matrices, such that $UU^\dagger = I$, i.e. $detU = \exp(i\alpha)$.

4) The special unitary group $SU(n)$ of the same matrices with additional condition $detU = 1$, i.e. $\alpha = 0$.

5) The orthogonal group $O(n)$ of $n \times n$ real matrices such that $R^T = R^{-1}$.

6) The special orthogonal group $SO(n)$ with the condition that $detR = 1$.

Definition.

The number of elements of the group G is called the **order** of G. The order of group may be finite, infinite but enumerable, or even infinite non-enumerable. In physical applications, as a rule, groups are either finite, or Lie groups - a special kind of non-enumerable infinite groups. This special class of groups will be studied in details later on.

Definition.

The group is called **Abelian group** if for all pairs of elements $\gamma_1, \gamma_2 \in G$ it is true that $\gamma_1\gamma_2 = \gamma_2\gamma_1$, i.e. all elements of G commute. In applications just non-Abelian groups are the most interesting.

2.2 Example: coordinate transformations in 3-dim Euclidean space.

Now we will consider the specific group of coordinate transformations in 3-dimensional space. This consideration will illustrate the general formulas of the previous Subsection, and also it is useful for applications both in 3-dimensional space and in 4-dimensional space-time. This group of transformations conserves angles between lines and distances between points in 3-dimensional space, including rotations and translations.

1. **3-dim rotations.**

Rotations T transforms coordinates $\vec{r} = (x, y, z)$ of the arbitrary point P to the new coordinates $\vec{r}' = (x', y', z')$ of the same point P. This is the so called passive point of view: we keep the point P fixed, but coordinate system rotates. One can check that, independently on the point P, the matrix $R(\gamma)$ exists such that

$$\vec{r}' = R(\gamma)\vec{r}.$$

For example, rotation γ for an angle θ about the axis $\vec{Ox}$:

$$\begin{aligned} x' &= x; \\ y' &= y\cos\theta + z\sin\theta; \\ z' &= -y\sin\theta + z\cos\theta \end{aligned}$$

corresponds to the matrix

$$R(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}; \tag{3}$$

which satisfies the condition $R^T(\gamma) = R^{-1}(\gamma)$ and does not change all scalar products:

$$\vec{r}_1 \cdot \vec{r}_2 = \vec{r}_1^T \cdot \vec{r}_2 = (\vec{r}_1^T R^T)(R\vec{r}_2) = \vec{r}_1' \cdot \vec{r}_2' \tag{4}$$

It is obvious from the orthogonality condition that $\det R(\gamma)$ may take on two values only: $+1$ or -1. The first class of rotations is called "proper" rotations, and the second - "improper". It is clear that rotations of a rigid body are always proper rotations. The "rotation" $\vec{r}' = -\vec{r}$ is an example of improper rotation. Arbitrary improper rotation can be represented as a product of the standard improper rotation $-I$ and some proper rotation. The product $\gamma_1\gamma_2$ of two rotations (γ_1 after γ_2,) gives also rotation but with the matrix

$$R(\gamma_1\gamma_2) = R(\gamma_1)R(\gamma_2). \tag{5}$$

The matrix $R(\gamma_1\gamma_2)$ has the same orthogonality properties as matrices $R(\gamma_1)$ and $R(\gamma_2)$. Matrices $R(\gamma)$ form the group of rotations $O(3)$, since all four group axioms are fulfilled. Matrices of proper rotations also form the group $SO(3)$.

2. **3-dim translations.**

Let us suppose now that mutually orthogonal Cartesian axes Ox, Oy, Oz were first rotating about some axis through the point O, and then translating O to O' along the vector $-\vec{t}(\gamma)$. In R^3 any two sets of Cartesian axes can be related in this way. Then we can write:

$$\vec{r}' = R(\gamma)\vec{r} + \vec{t}(\gamma) \tag{6}$$

where two terms in the r.h.s. represent two parts of a single transformation of coordinates γ. Therefore, it is convenient to rewrite (6) as

$$\vec{r}' = \langle R(\gamma)|\vec{t}(\gamma)\rangle\vec{r} \tag{7}$$

defining the composite operator $\langle R(\gamma)|\vec{t}(\gamma)\rangle$. It is useful to note here that there exist symmetry operations in which the combined rotations and translations leave the crystal lattice invariant but separate rotational and translational parts do not give such invariance.

Two successive transformations $\vec{r}' = \langle R(\gamma_2)|\vec{t}(\gamma_2)\rangle\vec{r} = R(\gamma_2)\vec{r}+\vec{t}(\gamma_2)$ and then $\vec{r}'' = \langle R(\gamma_1)|\vec{t}(\gamma_1)\rangle\vec{r}' = R(\gamma_1)\vec{r}' + \vec{t}(\gamma_1)$, give the resulting transformation:

$$\begin{aligned} \vec{r}'' &= \langle R(\gamma_1\gamma_2)|\vec{t}(\gamma_1\gamma_2)\rangle\vec{r} = R(\gamma_1)R(\gamma_2)\vec{r} + [R(\gamma_1)\vec{t}(\gamma_2) + \vec{t}(\gamma_1)] \\ &= \langle R(\gamma_1)R(\gamma_2)|R(\gamma_1)\vec{t}(\gamma_2) + \vec{t}(\gamma_1)\rangle. \end{aligned} \tag{8}$$

This product of transformations always satisfies the associative group axiom. The inverse to the element $\langle R(\gamma)|\vec{t}(\gamma)\rangle$ of the group is:

$$\vec{r} = R^{-1}(\gamma)\vec{r}' - R^{-1}(\gamma)\vec{t}(\gamma) = \langle R(\gamma)|\vec{t}(\gamma)\rangle^{-1}\vec{r} = \langle R^{-1}(\gamma)| - R^{-1}(\gamma)\vec{t}(\gamma)\rangle\vec{r}. \tag{9}$$

It is easily verified that

$$\langle R(\gamma_1\gamma_2)|\vec{t}(\gamma_1\gamma_2)\rangle^{-1} = \langle R(\gamma_2)|\vec{t}(\gamma_2)\rangle^{-1}\langle R(\gamma_1)|\vec{t}(\gamma_1)\rangle^{-1} \tag{10}$$

with inverse order of factors in the r.h.s. Transformations with $\vec{t}(\gamma) = \vec{0}$ are pure rotations, and transformations with $R(\gamma) = I$ are pure translations. The unity element of the group is:

$$E = \langle I|\vec{0}\rangle \tag{11}$$

2.3 Invariance of Schrödinger equation.

In Quantum Mechanics the quantum Hamiltonian of a physical system plays two important roles. At first, according to the stationary Schrödinger equation

$$H\Psi = E\Psi, \tag{12}$$

where by E the allowed energy eigenvalues of the Hamiltonian are denoted. At second, the same operator H describes the time evolution of the system via non-stationary Schrödinger equation

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{13}$$

It is natural to expect that one can know a lot about the physical system from investigation of the set of transformations which leave the Hamiltonian invariant. Actually, this is one of the main and one of the most creative functions of group theory in Quantum Mechanics.

For simplicity we will consider here the Schrödinger equation for one particle without internal degrees of freedom like spin etc., i.e. the typical Hamiltonian has the form:

$$H(\vec{r}) = -\frac{\hbar^2}{2m}\Delta^{(3)} + V(\vec{r}); \quad \Delta^{(3)} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{14}$$

where $V(\vec{r})$ is a potential.

Let $H(\langle R(\gamma)|\vec{t}(\gamma)\rangle\vec{r})$ be the operator which is obtained from $H(\vec{r})$ by substitution of $\vec{r}' = \langle R(\gamma)|\vec{t}(\gamma)\rangle\vec{r}$ instead of $\vec{r}$. In general, this new operator does not coincide with the initial $H(\vec{r})$, but for some specific class of transformations γ the Hamiltonian H is invariant. Thus, the Hamiltonian with Coulomb potential is invariant under all rotations, but it is not invariant under translations.

Theorem.

The coordinate transformations, after which the Hamiltonian is not changed, form a group. This group is called usually as the group of the Schrödinger equation or, equivalently, the invariance group of the Hamiltonian.

The proof of the Theorem is very simple: it is necessary only to check that all four group axioms are satisfied. The properties of coordinate transformations, considered above, give necessary tool for this proof.

By definition, a scalar function is a function which takes a value at each point without any dependence on the choice of coordinate system, i.e. it does not depend on the coordinates describing the point. This notion is suitable for discussion now, since the wave function Ψ of the scalar particle is scalar function. Let us denote the wave function in one coordinate system $\vec{O}x, \vec{O}y, \vec{O}z$ of R^3 as $\Psi(\vec{r})$, and the same scalar function but in anothe coordinate system $\vec{O}'x', \vec{O}'y', \vec{O}'z'$ as $\Psi'(\vec{r}')$. If $\vec{r}$ and $\vec{r}'$ describe the same physical point P, scalar character of Ψ means that

$$\Psi'(\vec{r}') = \Psi(\vec{r}). \tag{15}$$

Let us denote by T the transformation which links $\vec{r}$ and $\vec{r}'$, i.e. $\vec{r}' = \langle R(\gamma)|\vec{t}(\gamma)\rangle\vec{r}$. Thus, we have the constructive prescription for Ψ' from given Ψ :

$$\Psi'(\vec{r}') = \Psi(\langle R(\gamma)|\vec{t}(\gamma)\rangle^{-1}\vec{r}'), \tag{16}$$

where $\langle R(\gamma)|\vec{t}(\gamma)\rangle$ belongs to the invariance group of the Hamiltonian H.

We will introduce the special notation for this prescription $\Psi' = Q(\gamma)\Psi$, i.e.:

$$(Q(\gamma)\Psi)(\vec{r}) = \Psi(\langle R(\gamma)|\vec{t}(\gamma)\rangle^{-1}\vec{r}) \tag{17}$$

The scalar operators $Q(\gamma)$ have the following properties:

a) $Q(\gamma_1) = Q(\gamma_2)$ only if $\gamma_1 = \gamma_2$. It is important that $Q(\gamma_1) = Q(\gamma_2)$ for all functions $\Psi(\vec{r})$.

b) $Q(\gamma)$ is a linear operator, that is $Q(\gamma)\{\alpha\phi + \beta\psi\} = \alpha Q(\gamma)\phi + \beta Q(\gamma)\psi$ for arbitrary functions $\phi, \quad \psi$ and arbitrary complex numbers α, β.

c) $Q(\gamma)$ are unitary operators in the Hilbert space L^2 with scalar product, defined by:

$$(\phi, \psi) = \int \phi^\star(\vec{r})\psi(\vec{r})d^3r, \tag{18}$$

i.e.

$$(Q(\gamma)\phi, Q(\gamma)\psi) = (\phi, \psi). \tag{19}$$

The proof is rather straightforward and useful for home task.

d) For any pair of transformations γ_1, γ_2

$$Q(\gamma_1\gamma_2) = Q(\gamma_1)Q(\gamma_2). \tag{20}$$

This property also follows directly from the definition of $Q(\gamma)$.

e) The operators $Q(\gamma)$, which correspond to coordinate transformations γ from the group of Schrödinger equation, form the group, which is isomorphic to the group of Schrödinger equation. This can be proved by inspection of four group axioms.

f) For arbitrary transformation γ from the group of Schrödinger equation $Q(\gamma)$ commutes with the Hamiltonian H:

$$Q(\gamma)H(\vec{r}) = H(\vec{r})Q(\gamma). \tag{21}$$

Definition.

We say that d linearly independent functions $\psi_1(\vec{r}), \psi_2(\vec{r}), ..., \psi_d(\vec{r})$ form a basis of a $d-$dimensional matrix representation D of G, if for each coordinate transformation γ of G the following property is fulfilled:

$$Q(\gamma)\psi_n(\vec{r}) = \sum_1^d D(\gamma)_{mn}\psi_m(\vec{r}), \quad n = 1, 2, ..., d, \tag{22}$$

i.e. the function ψ_n is transformed as the $n-$th row of the representation D. We stress the unusual order of indices in (22), which ensures the group property for

every product $\gamma_1\gamma_2$:

$$\begin{aligned} Q(\gamma_1\gamma_2)\psi_n(\vec{r}) &= Q(\gamma_1)Q(\gamma_2)\psi_n(\vec{r}) = Q(\gamma_1)\sum_1^d D(\gamma_2)_{mn}\psi_m(\vec{r}) = \\ &= \sum_1^d D(\gamma_2)_{mn}Q(\gamma_1)\psi_m(\vec{r}) = \sum_1^d\sum_1^d D(\gamma_2)_{mn}D(\gamma_1)_{pm}\psi_p(\vec{r}) = \\ &= \sum_1^d D(\gamma_1\gamma_2)_{pn}\psi_p(\vec{r}). \end{aligned} \tag{23}$$

Theorem.

If the linearly independent eigenfunctions $\psi_i(\vec{r});\ i = 1, 2, ..., d$ satisfy the stationary Schrödinger equation with the same $d-$fold degenerate energy E, they form a basis for a $d-$dimensional representation of the invariance group G of the Schrödinger equation.

Proof. If $\psi_1(\vec{r}), \psi_2(\vec{r}), ..., \psi_d(\vec{r})$ is a set of linearly independent eigenfunctions of $H(\vec{r})$ with the same eigenvalue En :

$$H(\vec{r})\psi_n(\vec{r}) = E\psi_n(\vec{r}); \quad n = 1, 2, ..., d, \tag{24}$$

and any other eigenfunction of $H(\vec{r})$ with the same energy E is the linear combination of these $\psi_1(\vec{r}), \psi_2(\vec{r}), ..., \psi_d(\vec{r})$, then from Eq.(21) one obtains for arbitrary transformation $\gamma \in G$:

$$H(\vec{r})\{Q(\gamma)\psi_n(\vec{r})\} = Q(\gamma)\{H(\vec{r})\psi_n(\vec{r})\} = E\{Q(\gamma)\psi_n(\vec{r})\}. \tag{25}$$

Therefore, $Q(\gamma)\psi_n(\vec{r})$ is also an eigenfunction of $H(\vec{r})$ with the same energy E. Thus, $Q(\gamma)\psi_n(\vec{r})$ may be written as:

$$Q(\gamma)\psi_n(\vec{r}) = \sum_1^d D(\gamma)_{mn}\psi_m(\vec{r}),\ n = 1, 2, ..., d \tag{26}$$

Up to now, $D(\gamma)_{mn}$ were simply some coefficients, which depend explicitly on m, n and γ. These coefficients generate $d \times d$ matrices $D(\gamma)$. It is easy to show that for arbitrary transformations γ_1, γ_2

$$D(\gamma_1\gamma_2)_{mn} = \sum_1^d D(\gamma_1)_{mp}D(\gamma_2)_{pn}, \tag{27}$$

demonstrating that the matrices $D(\gamma)$ form actually a representation of the group G. Finally, one obtains:

$$Q(\gamma_1)Q(\gamma_2)\psi_n(\vec{r}) = \sum_1^d \sum_1^d D(\gamma_1)_{mp} D(\gamma_2)_{pn} \psi_m(\vec{r}). \tag{28}$$

This statement means that for each specific energy eigenvalue, a specific representation of the invariance group of Schrödinger equation exists.

2.4 Subgroups and classes.

Definition.

By definition, subgroup is a subset $\tilde{G}$ of a group G, if it itself forms a group with the same multiplication rule as in G. Formally speaking, G is also a subgroup of G. But we will consider so called *proper subgroups* which do not coincide with G. Analogously, The identity element E also form the formal subgroup itself. By the way, the identity E must be an element of each subgroup of G. The ratio g/s of orders of group G and its subgroup $\tilde{G}$, respectively, must be an integer number.

Now the useful criterium when the subset $\tilde{G}$ of G will be a subgroup can be formulated.

Theorem.

If a subset $\tilde{G}$ of a group G is such that for any two elements $S, S' \in \tilde{G}$ their product SS' also belongs to $\tilde{G}$, the subset $\tilde{G}$ is called a subgroup of G.

Proof. It is necessary to check all four group axioms. The associativity is fulfilled automatically in $\tilde{G}$. Taking $S' = S$ above one obtains that $S'S^{-1} = E$ belongs to $\tilde{G}$. Now $S' = E$ gives inverse element $S'S^{-1} = S^{-1}$. Finally, since $S^{-1} \in \tilde{G}$, one obtains that $S'(S^{-1})^{-1} = S'S \in \tilde{G}$, i.e. the first group axiom is also fulfilled.

Theorem (The Rearrangement Theorem).

For arbitrary fixed element γ' of a group G, the sets $\{\gamma'\gamma;\, \gamma \in G\}$ and $\{\gamma\gamma';\, \gamma \in G\}$ both contain each element of G one and only one time.

It means, for example, that for finite group one obtains the same group with different order of elements.

In Group Theory, class is a special type of sets of elements of G. Due to its special properties, class will play important role in representation theory.

At first, it is necessary to define conjugate elements of a group.

Definition.

An element γ' of a group G is called conjugate element to another element γ of G if such element X of G exists that

$$\gamma' = X\gamma X^{-1}. \tag{29}$$

It is clear that if γ' is conjugate to γ, then γ is conjugate to γ' with $X \to X^{-1}$. If γ and γ' both are conjugate to γ'', they are mutually conjugate. Therefore, we must consider a set of mutually conjugate elements of G.

Definition.

A set of mutually conjugate elements of the group G is called a conjugacy class or simply a class.

Starting from an arbitrary element γ of G, one can construct the class by means of $X\gamma X^{-1}$ for all $X \in G$. Of course, this class contains also γ itself (for $X = E$).

Properties of classes:

a) The identity element E forms the class itself, since for any $X \in G \quad XEX^{-1} = E$.

b) For Abelian group G every element forms a class itself, since $X\gamma X^{-1} = XX^{-1}\gamma = \gamma$.

c) Each element of group G belongs to some class, since one can take $X = E$.

d) No element of G belongs to two different classes of G. Indeed, if these two classes differ by elements γ' and γ'', then these two elements are also mutually conjugate, i.e. they can not belong to different classes.

2.5 Normal subgroup, factor group.

Our task now is to prepare construction of factor groups, which are important in different branches of theoretical physics.

Definition.

A subgroup $\tilde{G}$ of G is called an invariant subgroup (or normal subgroup, or normal divisor) if

$$XSX^{-1} \in \tilde{G} \tag{30}$$

for every $S \in \tilde{G}$ and every $X \in G$. It is clear that a close interrelations between invariant subgroups and classes exist.

Theorem.

A subgroup $\tilde{G}$ of a group G is an invariant subgroup if and only if $\tilde{G}$ consists of an arbitrary number of complete classes of G.

Proof. Let $\tilde{G}$ is an invariant subgroup. Then for any $S \in \tilde{G}$ and for any T of the same class as S, such $X \in G$ exists that $\gamma = XSX^{-1}$. Then due to (30) $\gamma \in \tilde{G}$ also. Thus the whole class of G lies in $\tilde{G}$.

Vice versa, let $\tilde{G}$ consists of the complete classes. Let us take arbitrary $S \in \tilde{G}$. Then all XSX^{-1} for all $X \in G$ form the class containing S, which belonged to $\tilde{G}$. Therefore, $XSX^{-1} \in \tilde{G}$ for all $S \in \tilde{G}$ and $X \in G$. Thus, $\tilde{G}$ is an invariant subgroup of G.

This Theorem gives very simple prescription to determine whether subgroup $\tilde{G}$ is invariant subgroup of G or not, for the cases when the classes of G are known.

Definition.

Let $\tilde{G}$ is a subgroup of G. For any fixed element $\gamma \in G$, which may belong or not belong to $\tilde{G}$, the set of elements of the form $S\gamma$, with S varying over the whole $\tilde{G}$, will be called the right coset of $\tilde{G}$ with respect to γ, and will be denoted by $\tilde{G}\gamma$. Similarly, the left coset $\gamma\tilde{G}$ is defined. It is important that, in general, $\gamma\tilde{G} \neq \tilde{G}\gamma$.

Theorem.

The properties of cosets.

a) If $\gamma \in \tilde{G}$, then $\tilde{G}\gamma = \tilde{G}$ due to Rearrangement Theorem.

b) Every element γ of G is a member of some right coset, since $\gamma = E\gamma$, and $E \in \tilde{G}$.

c) Any two elements $S\gamma$ and $S'\gamma$ of $\tilde{G}\gamma$ are different, provided that $S \neq S'$.

d) If γ does not belong to $\tilde{G}$, then $\tilde{G}\gamma$ is not a subgroup of G, since if $\tilde{G}\gamma$ would be a subgroup, such S must exist that $S\gamma = E$, i.e. $\gamma = S^{-1}$ would belong to $\tilde{G}$.

e) Two right cosets of $\tilde{G}$ are either identical or have no common elements.

f) If $\gamma' \in \tilde{G}\gamma$, then $\tilde{G}\gamma' = \tilde{G}\gamma$. Very important property: all members of coset are equivalent, any member of coset represents this coset.

g) If G is a finite group of order g, and $\tilde{G}$ is of order s, then the number of right cosets is g/s.

Theorem.

In order the right and left cosets of $\tilde{G}$ would be identical $\tilde{G}\gamma = t\tilde{G}$ for all $\gamma \in G$, it is necessary and sufficient that $\tilde{G}$ is an invariant group.

Let us take an invariant subgroup $\tilde{G}$ of G. There is variety of right cosets of $\tilde{G}$. We are not interested now in the internal structure of these cosets. The main object of this subsection is a set of right cosets and its structure.

Definition.

Let us define the product of two right cosets $\tilde{G}\gamma_1$ and $\tilde{G}\gamma_2$ according to the following rule:

$$\tilde{G}\gamma_1 \cdot \tilde{G}\gamma_2 = \tilde{G}(\gamma_1\gamma_2) \tag{31}$$

This definition is self consistent: one may take arbitrary another representatives of cosets in the l.h.s. of (31) instead of γ_1, γ_2. The resulting coset in the r.h.s. will be the same.

Theorem.

The right cosets of an invariant subgroup $\tilde{G}$ of G give a group with group multiplication law given by (31). This group will be called a factor group with a standard notation $G/\tilde{G}$.

2.6 Direct and semi-direct products.

This construction looks rather formal, but it has a lot of applications in different physical situations. Let us consider two groups G_1 with elements $\gamma_1 \in G_1$ and G_2 with elements $\gamma_2 \in G_2$. Let us construct now a variety of pairs (γ_1, γ_2) with a multiplication rule:

$$(\gamma_1, \gamma_2)(\gamma_1', \gamma_2') \equiv (\gamma_1\gamma_1', \gamma_2\gamma_2'), \tag{32}$$

where multiplications in the r.h.s. are defined by group multiplications in G_1 and G_2.

Theorem.

Elements (γ_1, γ_2) give the group with the group multiplication rule (32). This group is called as a direct product $G_1 \bigotimes G_2$.

It is easy to check that all four group axioms are fulfilled for $G_1 \bigotimes G_2$. Let us list some important properties of the direct product:

1) The direct product $G_1 \bigotimes G_2$ includes the subgroup with elements (γ_1, E_2), where $\gamma_1 \in G_1$ and E_2 is the identity element of G_2. This subgroup is isomorphic to G_1 itself: $\Phi((\gamma_1, E_2)) = \gamma_1$.

2) The analogous property is true for (E_1, γ_2).

3) Elements of the subgroups in 1) and 2) commute with each other:

$$(\gamma_1, E_2)(E_1, \gamma_2) = (E_1, \gamma_2)(\gamma_1, E_2) = (\gamma_1, \gamma_2). \tag{33}$$

4) These subgroups of the direct product have the only common element $(E_, E_2)$.

5) An arbitrary element of $G_1 \bigotimes G_2$ can be represented as $(\gamma_1, \gamma_2) = (\gamma_1, E_2)(E_1, \gamma_2)$.

The natural generalization of the direct product is called "the group of direct product". This group must be **isomorphic** to the direct product.

Theorem.

If the group G' includes two subgroups G'_1 and G'_2, such that the following properties are fulfilled:

– for arbitrary $\gamma'_1 \in G'_1$ and $\gamma'_2 \in G'_2$ they commute: $[\gamma'_1, \gamma'_2] = 0$;

– $E \in G'$ is the only common element of G'_1 and G'_2;

– arbitrary element $\gamma' \in G'$ can be written as a product $\gamma' = \gamma'_1 \gamma'_2$,

then the group G' is isomorphic to the direct product $G'_1 \bigotimes G'_2$.

Example.

The rotation group $O(3)$ is isomorphic to the direct product $SO(3) \bigotimes G'_2$, where G'_2 is a multiplicative group with two elements - $G'_2 = \{1_3, -1_3\}$ - identity and full inversion.

Let us notice the particular case, when both groups G'_1, G'_2 in the direct product are isomorphic to the same group G. This kind of direct product is used regularly in the theory of elementary particles. By the way, the direct product $G \bigotimes G$ includes the diagonal subgroup with elements (T, T).

One more remark concerns the first condition of the Theorem above. It can be replaced by another condition: both groups G'_1 and G'_2 are invariant subgroups of G'.

Definition.

If the first condition in the Theorem above is weakened so that G'_1 is invariant subgroup of G', but G'_2 is a subgroup (not necessarily invariant), then G' is called the semi-direct product of G'_1 and G'_2, i.e. $G' = G'_1 \odot G'_2$.

2.7 Homomorphic and isomorphic mapping.

Let us consider two groups G and G'. Let Φ is a mapping of G onto G', i.e. each element γ of G is connected with some element γ' of G' : $\gamma' = \Phi(\gamma)$. Let us suppose that each element of G' is the image of at least one element of G. If this correspondence is one-to-one, the inverse mapping also exists: $\gamma = \Phi^{-1}(\gamma')$.

Definition.

If Φ is a mapping of group G onto group G' such that for all $\gamma_1, \gamma_2 \in G$

$$\Phi(\gamma_1)\Phi(\gamma_2) = \Phi(\gamma_1\gamma_2), \tag{34}$$

this mapping Φ is called a homomorphic. The well known example of homomorphic mapping was already mentioned above.

Definition.

A homomorphic mapping of group G onto group of non-singular $d \times d$ matrices $D(T)$ with matrix multiplication as a group operation creates a $d-$ dimensional representation D of G by the group of matrices $D(\gamma)$.

In general, the mapping above is not one-to-one. But if it is one-to-one it is called "isomorphic mapping". The corresponding representations are called "faithful". In this case the mapping ϕ^{-1} also exists and it is isomorphic as well. Even if isomorphic groups have different nature of their elements, their structures are similar: they have the same structures of subgroups, cosets, classes etc etc. And what is much more important: they have the same representations.

Definition.

If Φ is a homomorphic mapping G onto G', the set of elements T of G, which are mapped onto identity of G' : $\Phi(\gamma) = E'$, is called kernel of the mapping Φ.

Theorem.

If K is a kernel of homomorphic mapping G onto G', then

a) K is invariant subgroup of G.

b) Each element of the right coset $K\gamma$ are mapped onto the same element $\Phi(\gamma)$ of G', and the corresponding mapping κ such that $\kappa(K\gamma) = \Phi(\gamma)$ is a one-to-one mapping of the factor group G/K onto G'.

c) This mapping κ is an isomorphic mapping.

The special case of isomorphism when G' coincides with G is called automorphism. For example, each element $X \in G$ defines the automorphism

$$\Phi_X(\gamma) = X\gamma X^{-1} \tag{35}$$

which is called inner automorphism. All other automorphisms are called outer automorphisms.

3 Lie groups and their representations.

3.1 Linear Lie groups.

Lie groups contain three different mathematical structures:

1) They obey four group axioms and therefore they have group structure.

2) Elements of Lie groups span a topological space, and therefore they have topological structure.

3) Elements of Lie group give an analytical manifold, and therefore they have analytical structure.

This is a reason why Lie groups can be defined in different ways, which are equivalent. Since we are interested in physical applications of Lie groups, we may restrict ourselves with a particular case of Lie groups - linear Lie groups. Just in this case the definitions are rather simple and straightforward being based mainly on the group structure. The main idea is that any Lie group has a non-countable number of elements, which lie near the identity element. The structure of this region close to identity is defined by the corresponding Lie algebra, and in turn, this structure defines the structure of the whole group. So, the region near identity must be parametrized analytically, and the notion of distance has to be defined in order to have notion "near" and "far". This is the main difficulty of general consideration of Lie groups. However, the advantage of Lie groups of physical interest is that are linear: they have at least one faithful finite-dimensional representation, which can be used to define the distance between group elements.

Definition.

A group G is called a $n-$dimensional linear Lie group if it satisfies the following four conditions:

A) G has at least one faithful finite-dimensional representation D. Let us denote its dimensionality by m. Then one can define the distance $d(\gamma, \gamma')$ between any two elements γ, γ' of G by metric:

$$d(\gamma, \gamma') = \{ \sum_{j,k=1}^{m} |D(\gamma)_{jk} - D(\gamma')_{jk}|^2 \}^{1/2}, \tag{36}$$

with the usual properties:

$$d(\gamma,\gamma') = d(\gamma',\gamma);$$
$$d(\gamma,\gamma) = 0;$$
$$d(\gamma,\gamma') > 0 \qquad if \qquad \gamma \neq \gamma';$$
$$d(\gamma,\gamma'') \leq d(\gamma,\gamma') + d(\gamma',\gamma'').$$

This definition of metric means that the group has topology of the m^2-dimensional complex Euclidean space C^{m^2}. A small neighbourhood of identity can be defined as a sphere M_δ of radius δ with center at identity E, i.e. the set of elements $\gamma \in G$ such that $d(\gamma, E) < \delta$.

B) A small $\delta > 0$ must exist such that all elements $\gamma \in G$, which belong to the sphere M_δ, can be parametrized by n real parameters $x_1, x_2, ..., x_n$ (with one-to-one correspondence), the identity E will have $x_1 = x_2 = ... = x_n = 0$.

C) And vice versa, such small $\eta > 0$ must exist that every point in R^n, for which $\sum_1^n x_j^2 < \eta^2$, corresponds to some element Tin M_δ. This set of points T (denoted by R_η) is a subset of M_δ.

D) All matrix elements of $D(x_1, x_2, ..., x_n) \equiv D(\gamma(x_1, x_2, ..., x_n))$ are analytic functions of its arguments for all x_j from R_η. By analyticity we mean an opportunity to expand the function in a power series in $x_1 - a_1, x_2 - a_2, ..., x_n - a_n$ for arbitrary point $a_1, a_2, ..., a_n$ from R_η. This means also that all partial derivatives of $D(T)_{jk}$ over x_l must exist in R_η.

Let us introduce n matrices $m \times m$:

$$(X_p)_{jk} = \frac{\partial D_{jk}}{\partial x_p}_{|x_1=x_2=...=x_n=0}. \tag{37}$$

These matrices form the basis for $n-$dimensional real vector space (but matrix elements D may be complex also).

Let us remark that the parametrization by $x_1, x_2, ..., x_n$ was introduced in some region M_δ around the identity E only. Of course, it might be extended as parametrization $y_1, y_2, ..., y_n$ also in wider region, but in this case other properties of linear Lie group for $y_1, y_2, ..., y_n$ may be violated.

The group operations of multiplication and inversion are analytical for linear Lie groups due to the following Theorem. Let us suppose that U is a subset of R_η such that if γ, γ' are from U then $\gamma'' = \gamma\gamma'$ is from R_η. Let coordinates of $\gamma, \gamma', \gamma''$ are $(x_1, x_2, ..., x_n), (x'_1, x'_2, ..., x'_n), (x''_1, x''_2, ..., x''_n)$, correspondingly. then $x''_j = f_j(x_1, x_2, ..., x_n; \quad x'_1, x'_2, ..., x'_n)$

Theorem.

Functions $f_j(x_1, x_2, ..., x_n; \quad x'_1, x'_2, ..., x'_n)$ above are analytic functions of their arguments for all γ, γ' in U. The analogous theorem is true for the elements γ^{-1}.

Example.

The multiplication group of real numbers $t \neq 0$. The obvious one-dimensional representation can be chosen as $D(t) = t$, so the condition A) is satisfied, and the metric is given by $d(t, t') = |t - t'|$. Let's take $\delta = 1/2$, then $M_\delta = (1/2, 3/2)$. The convenient parametrization for $t \in M_\delta$ is $t = \exp(x_1)$. The identity corresponds to $x_1 = 0$. Exponential $D(x_1)$ is obviously analytical function of x_1. Thus, the group is the linear Lie group of dimensionality 1, and $X_1 = 1$. In this case, the parametrization x_1 can be extended to all $t > 0$, but if $t < 0$, it can be written as $(-1)\exp(x_1)$.

Example.

Let us consider the group $O(2)$ of all real orthogonal 2×2 matrices A. Its subgroup with $detA = 1$ gives $SO(2)$. In this example $D(A) = A$ gives necessary faithful representation. Since $A^T A = AA^T = 1$ only two solutions exist:

1) $A_{11} = A_{22}$ and $A_{12} = -A_{21}$. In this case, $detA = +1$ and $d(A, 1) = 2(1 - A_{11})^{1/2}$.

2) $A_{11} = -A_{22}$ and $A_{12} = A_{21}$. Here $detA = -1$, and $d(A, 1) = 2$.

Choosing $\delta = 1/2$, condition B) requires to parametrize the set 1) only, and the set 2) is outside M_δ. The convenient parametrization is

$$A = D(A) = \begin{pmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{pmatrix}. \tag{38}$$

The point $x_1 = 0$ corresponds to identity $E = I$. The condition C) is satisfied since for each point in R^1 such that $|x_1| < \frac{\pi}{3}$ the matrix $A(x_1)$ lies in M_δ. Practically, this parametrization can be extended to the whole subgroup $SO(2)$. The condition D) is obviously satisfied also, therefore $O(2)$ and $SO(2)$ are linear Lie groups of dimension 1. The operator $X_1 = i\sigma_2$ in this case. The general conclusion is that for all $n \geq 2$ the groups $O(n), SO(n), U(n), SU(n)$ belong to the class of linear Lie groups with dimensions $1/2n(n-1), 1/2n(n-1), n^2, n^2 - 1$, respectively.

Example.

In the previous examples we considered groups of matrices, where condition A) is satisfied automatically. In the following example the initial group is not matrix group. We mean the group of coordinate transformations in Euclidean space R^3, which were studied in Section 2. In this case one can use a trick to

present the transformations $\langle R(\gamma)|\vec{t}(\gamma)\rangle$ as matrices

$$D(\langle R(\gamma)|\vec{t}(\gamma)\rangle) = \begin{pmatrix} R_{11} & R_{12} & R_{13} & t_1 \\ R_{21} & R_{22} & R_{23} & t_2 \\ R_{31} & R_{32} & R_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

such that

$$D(\gamma) = \begin{pmatrix} R(\gamma) & \vec{t}(\gamma) \\ 0 & 1 \end{pmatrix} \tag{40}$$

Then it is again easy to demonstrate that all four conditions of linear Lie group are satisfied. This group has dimension 6.

3.2 Connected components. Compact groups.

Definition.

Let us choose a maximal set of elements $\gamma \in G$, which can be obtained from each other by a continuous change of one (or more) matrix elements $D(\gamma)_{jk}$ of the faithful finite dimensional representation. This maximal set is called as a connected component of the group G.

Example.

In the group of real numbers ($t \neq 0$) with multiplicative rule of group composition two connected components can be found. The first, $t > 0$, is actually a subgroup (it includes identity $t = 1$), but the second, $t < 0$, is not a subgroup and is separated from the first by excluded point $t = 0$.

Theorem.

If the connected component of linear Lie group contains identity, it is invariant subgroup. This invariant subgroup is called connected subgroup in G. Each connected component of a linear Lie group is a (right) coset of connected subgroup.

In physical systems usually group G has a finite number of connected components. If G has only one connected component, this group is called the connected group.

Theorem.

A subset of points of a real or complex finite dimensional Euclidean space is compact if and only if it is closed and bounded.

In many cases it is useful to introduce instead of (or in addition to) parametrization by $x_1, x_2, ..., x_n$ around identity another parametrization

$y_1, y_2, ..., y_n$ which can be used over the whole connected component of linear Lie group.

Definition.

A linear Lie group with a finite number of connected components is compact if the parameters $y_1, y_2, ..., y_n$ vary in the closed finite intervals $y_j \in [a_j, b_j]$, $j = 1, 2, ..., n$.

Physically interesting non-compact Lie groups have unbounded set of matrix elements in $\mathcal{C}^{m^2}$. Since the set of matrix elements $D(\gamma)_{jk}$ of a linear Lie group are bounded if and only if a finite real number M exists such that $d(\gamma, E) < M$ for all $\gamma \in G$, the non-compact groups are rather simply recognized in practice. If the Lie group is compact, all its subgroups are also compact. But subgroups of non-compact Lie group can be either compact or non-compact. It is necessary to notice, that the theory of representations for compact linear Lie groups is essentially similar to the theory for the finite groups. The situation for non-compact groups is different.

Example.

The group of real numbers $t \neq 0$ with multiplicative group operation is obviously non-compact: its faithful representation is parametrized by unbounded set $t \in (-\infty, 0) \bigcup (0, \infty)$.

Groups $O(n), SO(n), U(n), SU(n)$ are compact, but physically important relativistic Lorentz and Poincare groups are non-compact.

3.3 Representations: basic notions.

Definition.

The group of non-singular $d \times d$ matrices $D(\gamma)$ with the matrix multiplication as the group operation forms a $d-$dimensional representation D of G, if it is an image of a homomorphic mapping of a group G. If the mapping is one-to-one, it is called faithful.

For all representations $D(E) = I$ and $D(\gamma^{-1}) = D^{-1}(\gamma)$. Every group has the trivial - identity - representation $D(\gamma) = I$ for all $\gamma \in G$.

If the group is a linear Lie group, it is necessary to add special condition that the homomorphic mapping is continuous. It means for a connected linear Lie group that the matrix elements $D(\gamma)_{jk}$ must be continuous functions of parameters $y_1, y_2, ..., y_n$, introduced in the previous Section.

Analogously to operators $P(\gamma)$ and the basis functions $\psi_n(\vec{r})$, which were introduced for the group of coordinate transformations in Euclidean space R^3,

it is useful to make the similar construction for an arbitrary group G. So, let us consider a $d-$dimensional representation D of G, and let $\psi_1, \psi_2, ..., \psi_d$ are the elements of the basis of a $d-$dimensional abstract complex inner product space, which we will call the carrier space V. Then let us define for every element $\gamma \in G$ the operator $\Phi(\gamma)$ such that:

$$\Phi(\gamma)\psi_n = \sum_1^d D(\gamma)_{mn}\psi_m. \tag{41}$$

Let us define also the operator $\Phi(\gamma)$ as a linear operator. From (41) it follows that for all $\gamma_1, \gamma_2 \in G$

$$\Phi(\gamma_1\gamma_2) = \Phi(\gamma_1)\Phi(\gamma_2), \tag{42}$$

i.e. these operators form a group, and the homomorphic mapping from G onto this group exists. The complex $(\Phi(\gamma), V)$ is called module. It is necessary to notice that, in general, for a given representation, $\Phi(\gamma)$ are not unitary. If the basis ψ_n is orthonormal, (41) leads to

$$D(\gamma)_{mn} = (\psi_m, \Phi(\gamma)\psi_n) \tag{43}$$

for arbitrary $\gamma \in G$. In turn, arbitrary set of operators, which act on a $d-$dimensional inner product space and satisfy (42), produces, according to (43), a $d-$dimensional matrix representation. By the way, this is a best way to introduce infinite-dimensional representations of G.

Theorem.

If $D(\gamma)$ is the $d-$dimensional representation of G, and S is an arbitrary non-singular $d \times d$ matrix, then matrices

$$D'(\gamma) \equiv S^{-1}D(\gamma)S \tag{44}$$

also realize a $d-$dimensional representation of G, which we will call as equivalent representation .

It is clear that for $d = 1$ always $D'(\gamma) = D(T)$ for all $\gamma \in G$. Therefore, two arbitrary one-dimensional representations are either identical or are not equivalent. As for more interesting situation with $d \geq 2$, matrices $D'(\gamma)$ and $D(\gamma)$ are different. Nevertheless, equivalent representations are very close in their properties. In particular, the following theorem explains that equivalence (similarity) transformation corresponds to rearrangement of basis.

Theorem.

If $\psi_1, \psi_2, ..., \psi_d$ form a basis for a $d-$dimensional representation and if S is arbitrary non-singular $d \times d$ matrix, then the functions

$$\psi'_n(\vec{r}) = \sum_1^d S_{mn}\psi_m(\vec{r}) \tag{45}$$

give the basis for the equivalent representation D' such that $D'(\gamma) = S^{-1}D(\gamma)S$, and $\Phi(\gamma)\psi'_n = \sum_1^d D'(\gamma)_{mn}\psi'_m$.

Definition.

The representation, for which all matrices $D(T)$ are unitary, is called the unitary representation.

Theorem.

For finite or compact group G all representations are equivalent to unitary representations .

Later on the notions of simple and semi-simple groups will be introduced. Let us formulate in advance important statements concerning representations of these groups.

Theorem.

For non-compact simple Lie group no finite-dimensional unitary representations exist (up to trivial representation $D(\gamma) = I$ for all $\gamma \in G$).

But non-compact Lie groups which are not simple may have both unitary representations and representations not equivalent to unitary. For example, the group of coordinate transformations $\langle R(\gamma)|\vec{t}(\gamma)\rangle$ of R^3 has both types of representations . In particular, if representation depends on rotations only - $D(\langle R(\gamma)|\vec{t}(\gamma)\rangle) = D(R(\gamma))$ - then finite-dimensional representations exist.

In physical applications the following Theorem plays important role.

Theorem.

If D and D' are equivalent representations , which are connected by similarity transformation

$$D'(\gamma) = S^{-1}D(\gamma)S, \tag{46}$$

and if D is unitary representation and S is unitary matrix, $D'(\gamma)$ is also unitary representation . In turn, if D and D' are equivalent unitary representations , the matrix S can be also chosen unitary.

3.4 Reducibility.

Let us consider the representation $D(\gamma)$ such that matrices for all $\gamma \in G$ can be written in the similar form:

$$D(\gamma) = \begin{pmatrix} D_{11}(\gamma) & D_{12}(\gamma) \\ 0 & D_{22}(\gamma) \end{pmatrix}, \tag{47}$$

where all blocks $D_{11}(\gamma)$, $D_{12}(\gamma)$, $D_{22}(\gamma)$ and the matrix 0 are matrices of dimensionalities $s_1 \times s_1$, $s_1 \times s_2$, $s_2 \times s_2$, $s_2 \times s_1$, respectively.

The product of two matrices of the form (47) has the same form:

$$D(\gamma_1)D(\gamma_2) = \begin{pmatrix} D_{11}(\gamma_1)D_{11}(\gamma_2) & D_{11}(\gamma_1)D_{12}(\gamma_2) + D_{12}(\gamma_1)D_{22}(\gamma_2) \\ 0 & D_{22}(\gamma_1)D_{22}(\gamma_2) \end{pmatrix}, \tag{48}$$

therefore matrices $D_{11}(\gamma)$ and $D_{22}(\gamma)$ themselves give the representations of the group G :

$$D_{11}(\gamma_1\gamma_2) = D_{11}(\gamma_1)D_{11}(\gamma_2); \quad D_{22}(\gamma_1\gamma_2) = D_{22}(\gamma_1)D_{22}(\gamma_2). \tag{49}$$

Definition.

If the representation of a group G for all elements T is equivalent to (47), it is called reducible representation . And vice versa, if the representation is not equivalent to (47), it is called irreducible .

Obviously, the process of reducing can be continued, up to the moment when all block matrices will become irreducible , and the matrix $D(T)$ will take the form:

$$D'(\gamma) = S^{-1}D(\gamma)S = \begin{pmatrix} D'_{11}(\gamma) & D'_{12}(\gamma) & D'_{13}(\gamma) & ... & D'_{1r}(\gamma) \\ 0 & D'_{22}(\gamma) & D'_{23}(\gamma) & ... & D'_{2r}(\gamma) \\ 0 & 0 & D'_{33}(\gamma) & ... & D'_{3r}(\gamma) \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & ... & D'_{rr}(\gamma) \end{pmatrix}$$

where all matrices at diagonal $D'_{kk}(\gamma)$ are irreducible .

Definition.

We call the representation completely irreducible if it is equivalent to the following block diagonal form:

$$D'(\gamma) = S^{-1}D(\gamma)S = \begin{pmatrix} D'_{11}(\gamma) & 0 & 0 & ... & 0 \\ 0 & D'_{22}(\gamma) & 0 & ... & 0 \\ 0 & 0 & D'_{33}(T) & ... & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & ... & D'_{rr}(T) \end{pmatrix}.$$

The important Theorem states:

Theorem.

Every reducible representation of a finite group or a compact Lie group is completely reducible . The analogous conclusion is true also for every reducible representation of a connected, non-compact, semisimple Lie group and for every unitary reducible representation of any other group.

Two Schur's theorems are known, and both are very important for physical applications of Group Theory.

First Schur's Lemma.

If all matrices of irreducible $d-$dimensional representation commute with the same $d \times d$ matrix M, then M is proportional to unity:

$$D(\gamma)M = MD(\gamma) \Rightarrow M = \lambda \cdot I \tag{52}$$

Second Schur's Lemma.

Let us suppose that for two irreducible representations D and D' of dimensionality d and d', respectively, such $d \times d'$ matrix M exist that:

$$D(\gamma)M = MD'(\gamma)$$

for all $\gamma \in G$. Then either $M = 0$, or $d = d'$ and $detM \neq 0$

The straightforward corollary of these Theorems concerns Abelian groups: every irreducible representation of an Abelian group is one-dimensional.

For physical applications the following theorem is very important.

Theorem.

Let D^p and D^q are two unitary irreducible representations of a compact Lie group of dimensionality d_p, d_q, respectively. Let these representations are not equivalent if $p \neq q$ and coincide if $p = q$. Then:

$$\int_G D^p(\gamma)^*_{jk} D^q(\gamma)_{st} d\gamma = \frac{1}{d_p} \delta_{pq} \delta_{js} \delta_{kt}$$

This is the so called Theorem of orthogonality of matrix representations .

4 Lie algebras.

4.1 Lie group and Lie algebra.

In the definition of linear Lie groups we used some special parametrization $x_1, x_2, ..., x_n$ of matrix representations of the group G in the neighbourhood of identity element $x_j = 0$ of G (or, equivalently, of unit matrix of the representation). In this section and further on we will use notations $\alpha_1, \alpha_2, ..., \alpha_N$ for real valued parameters instead of old $x_1, x_2, ..., x_n$ and also we will redefine operators X_i of (37) as follows:

$$X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a}_{|\alpha=0}. \tag{55}$$

The operators X_a are called generators of the group. To be more accurate, X_a are the representations of generators of the group, but below we will be interested in its unitary representations only. Just by this reason, we prefer redefinition (55) keeping X_a Hermitian operators.

We may choose an arbitrary parametrization in the neighbourhood of identity to go away from I. In particular, we may use an infinitesimal element

$$D(d\alpha) = I + id\alpha_a X_a, \tag{56}$$

and its arbitrary powers, which due to group multiplication property will give finite elements:

$$D(\alpha) = \lim_{k\to\infty} (1 + i\frac{\alpha_a}{k} X_a)^k = \exp(i\alpha_a X_a). \tag{57}$$

This exponential parametrization allows to write the elements of group (actually, of its representation) in terms of generators X_a of the group G. It is very promising since just the generators form a vector space, i.e. one can build their linear combinations with real coefficients. Practically, one can use the term generators in respect to arbitrary element of this linear vector space spanned by X_a.

Thus we have one parameter families of group elements:

$$W(\lambda) = \exp(i\lambda\alpha_a X_a), \tag{58}$$

with multiplication rule between members of the same family:

$$W(\lambda_1)W(\lambda_2) = W(\lambda_1 + \lambda_2). \tag{59}$$

A much more difficult question concerns the product of members of different families of the type (58), since in general,

$$\exp\left(i\alpha_a X_a\right)\cdot\exp\left(i\beta_a X_a\right)\neq\exp\left[i(\alpha_a+\beta_a)X_a\right]. \tag{60}$$

But it is clear that the multiplies at the l.h.s. of (60), being elements of the representation of the group G (at least, if we are close to identity), must lead to some another element of the same representation , which has the exponential form, i.e.

$$\exp\left(i\alpha_a X_a\right)\cdot\exp\left(i\beta_a X_a\right)=\exp\left(i\delta_a X_a\right) \tag{61}$$

for some set of δ_a.

How could we find numbers δ_a for given α_a, β_a ? Because we deal here with analytic functions, we may expand all terms in series and compare the terms with the same total powers of α_a and β_a with powers of δ_a. We will demonstrate now explicitly, that this is possible only if generators X_a satisfy some specific restriction.

We will act up to first nontrivial orders in variables. From (61) we have:

$$i\delta_a X_a=\ln\left(1+\exp\left(i\alpha_a X_a\right)\cdot\exp\left(i\beta_a X_a\right)-1\right), \tag{62}$$

and we will expand this equality by standard rules up to second order in α_a, β_a :

$$\begin{aligned} M &\equiv \exp\left(i\alpha_a X_a\right)\cdot\exp\left(i\beta_a X_a\right)-1= \\ &= \left(1+i\alpha_a X_a-\frac{1}{2}(\alpha_a X_a)^2+...\right)\cdot\left(1+i\beta_a X_a-\frac{1}{2}(\beta_a X_a)^2+...\right)-1= \\ &= i\alpha_a X_a+i\beta_a X_a-\alpha_a X_a\beta_b X_b-\frac{1}{2}(\alpha_a X_a)^2-\frac{1}{2}(\beta_a X_a)^2+... \end{aligned} \tag{63}$$

Therefore, $\ln(1+M)=M-\frac{1}{2}M^2+...$ gives:

$$\begin{aligned} i\delta_a X_a &= i\alpha_a X_a+i\beta_a X_a-\alpha_a X_a\beta_b X_b-\frac{1}{2}(\alpha_a X_a)^2-\frac{1}{2}(\beta_a X_a)^2+\frac{1}{2}(\alpha_a X_a+\beta_a X_a)^2...= \\ &= i\alpha_a X_a+i\beta_a X_a-\frac{1}{2}[\alpha_a X_a,\beta_b X_b]+..., \end{aligned} \tag{64}$$

where we took into account that X_a are operators which probably do not commute with each other. After rewriting (64) in a slightly different form we obtain:

$$[\alpha_a X_a,\beta_b X_b]=-2i(\delta_c-\alpha_c-\beta_c)X_c+...=i\varsigma_c X_c, \tag{65}$$

where ς is real, and omitted terms are of more than second order in α, β. The last equality has to be true for all values of α, β, and therefore

$$\varsigma_c = f_{abc}\alpha_a\beta_b \tag{66}$$

with some constant coefficients f_{abc}.

Thus the commutator of X_a must be reduced to a linear combination:

$$[X_a, X_b] = if_{abc}X_c, \tag{67}$$

where constants f_{abc} has to be antisymmetric in first two indices:

$$f_{abc} = -f_{bac}, \tag{68}$$

due to antisymmetry of commutators.

Now we can write that:

$$\delta_a = \alpha_a + \beta_a - \frac{1}{2}\varsigma_a + ... \tag{69}$$

If ς_a and higher order terms would vanish, we will obtain the trivial case (60).

Thus from the group multiplication operation for Lie groups, due to their smooth dependence on parameters, we derived the equation (67), which just defines the algebra of generators X_a. The commutator (67) in algebra plays the role similar to multiplication rule in the Lie group.

One may worry that the rule (67) will not be enough to satisfy (62) up to higher terms in α_a, β_a. The important statement is that it is enough, i.e. one can calculate δ_a with arbitrary accuracy in some neighbourhood of identity, using (67) without any modifications.

The constants f_{abc} are called the structure constants of the group, and they play very important role in the whole construction. The set of commutation relations (67) for different values of a, b is called the Lie algebra of the initial Lie group. Each representation of the Lie group gives the corresponding representation of Lie algebra. The important fact is that the structure constants do not depend on the chosen representation , since these constants were defined from the group multiplication operation (and smoothness on parameters) only. Analogously, one can transfer to the Lie algebra context the notions of equivalence, reducibility and irreducibility of representations .

One can notice, that for the Hermitian generators X_a

$$[X_a, X_b]^\dagger = -if^*_{abc}X_c = [X_b, X_a] = if_{bac}X_c = -if_{abc}X_c \tag{70}$$

the constants f_{abc} are real. Since we are interested here in groups with unitary representations , this means that we may deal with real structure constants from the very beginning.

It is clear from definition, that for Abelian group the structure constants $f_{abc} = 0$.

If H is a subgroup of the Lie group G, one may choose the generators of Lie algebra of G in such a way that the first p operators $X_1, X_2, ..., X_p$ form the Lie algebra of the subgroup H, and the rest $N - p$ fill the whole Lie algebra. Then due to definition of subgroup, its Lie algebra must be closed, i.e.

$$f_{ijk} = 0 \qquad for \qquad i, j = 1, 2, ..., p; \quad k = p + 1, p + 2, ..., N. \tag{71}$$

If H is invariant subgroup, one more restriction for structure constants must be satisfied:

$$f_{ikl} = 0 \qquad for \qquad i = 1, 2, ..., p; \quad k, l = p + 1, p + 2, ..., N. \tag{72}$$

To prove this statement one has to remind that for invariant subgroup

$$\gamma^{-1} S \gamma \in H$$

for arbitrary $S \in H$ and $\gamma \in G$. Then

$$S^{-1} \gamma^{-1} S \gamma \in H$$

is also true. If we rewrite this element of H in terms of generators of Lie subalgebra up to second order in parameters, we will obtain just the statement (72).

If G has no invariant subgroups (i.e. G is a simple group), no subset of the structure constants satisfy the restrictions (72) independently on the choice of basis among X_a.

If G has no Abelian invariant subgroups (i.e. G is a semisimple group), (72) can be fulfilled, but then the additional restriction $f_{ikl} = 0 \quad for \quad i, k, l = 1, 2, ..., p$ is forbidden.

In more general form, a Lie algebra is a real vector space with antisymmetric operation - multiplication - which belong to the same space but with one additional condition. The so called Jacobi identity must be true:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \tag{73}$$

The Jacobi identity is certainly satisfied for arbitrary matrix representation , since three matrices satisfy (73) automatically. But in general operatorial form this is an additional nontrivial condition.

One can notice that the Jacobi identity after rewriting in another form:

$$[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]] \tag{74}$$

can be treated as a natural generalization of the well known rule for calculation of triple commutator:

$$[X_a, X_b X_c] = [X_a, X_b] X_c + X_b [X_a, X_c]. \tag{75}$$

The problem to find all possible realizations of structure constants with fixed dimensionality N seems to be very difficult. Indeed, it means to find the general solution of the following nonlinear system of algebraic equations:

$$f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0; \tag{76}$$

$$f_{abc} = f_{bac}. \tag{77}$$

Besides the nonlinearity of (76), the difficulty is related with equivalence of many different sets of f_{abc}, which are connected by linear transformations with each other.

Let us analyze the problem for the lowest values of N.

1) $N = 1$. This case corresponds to one-parametric Abelian group.

2) $N = 2$. The general possibility for the case of two generators X_1, X_2 is given by:

$$[X_1, X_2] = iaX_1 + ibX_2, \tag{78}$$

with arbitrary real constants (structure constants) a, b. It can be splitted onto two qualitatively different cases:

a) $a = b = 0$, i.e. two-parametric Abelian group which can be described as a direct product of two systems of the type 1). As example, one can keep in mind the group of translations in two-dimensional plane:

$$x' = x + a; \quad y' = y + b.$$

b) $a \neq 0$. In this case one can choose suitable generators as the linear combinations:

$$X_1' = aX_1 + bX_2;$$
$$X_2' = \frac{1}{a} X_2.$$

Then the only nonvanishing commutator takes the form:

$$[X_1', X_2'] = iX_1'. \tag{79}$$

It means that the subgroup generated by X_1' is invariant (and Abelian) subgroup. Therefore, the group G with the Lie algebra (79) can not be semisimple.

3) $N-3$. In this case two different options (up to linear combinations of X_a) exist.

a) The first one:

$$[X_1, X_2] = iX_3; \quad [X_2, X_3] = iX_1; \quad [X_3, X_1] = iX_2, \tag{80}$$

which corresponds to the Lie group $O(3)$ with generators realized by differential operators:

$$X_1 = i(x\partial_y - y\partial_x); X_2 = i(y\partial_z - z\partial_y); X_3 = i(z\partial_x - x\partial_z).$$

b) The second option differs by some signs in r.h.s.:

$$[X_1, X_2] = iX_3; \quad [X_2, X_3] = -iX_1; \quad [X_3, X_1] = -iX_2, \tag{82}$$

corresponding to the Lie group $O(2,1)$ with generators:

$$X_1 = i(z\partial_y + y\partial_z); X_2 = i(x\partial_z + z\partial_x); X_3 = i(y\partial_x - x\partial_y).$$

4.2 Simple and semisimple Lie algebras.

We mentioned already the terms of simple and semisimple groups. In terms of Lie algebras, we have to introduce the definition of ideals in Lie algebras.

Definition.

If for subalgebra B with generators $X \in B$ of Lie algebra A with generators $Y \in A$ it is true that:

$$[X, Y] \in B, \tag{84}$$

then B is called ideal, or invariant subalgebra, in A, If $B \neq A$, H is called the proper ideal.

The simple Lie algebra does not contain proper ideals. The semisimple Lie algebra does not contain proper Abelian ideals.

Why semisimple algebras are of such importance ? The reason is in the following Theorem.

Theorem.

Algebra Lie A is semisimple if and only if A is a direct sum of ideals A_i, corresponding to simple Lie algebras.

4.3 Adjoint representation.

The structure constants of arbitrary Lie algebra give the tool for construction of some specific representation - adjoint representation of the Lie group, which plays very important role in the general theory of representations . The adjoint representation is defined by generators:

$$(T_a)_{bc} = -if_{abc}. \tag{85}$$

As was shown recently, the Jacobi identity (73) can be rewritten in terms of structure constants f_{abc} (see (76)). Just this relation leads to the commutation relations between generators (85) of adjoint representation :

$$[T_a, T_b] = if_{abc}T_c, \tag{86}$$

which are necessary to realize representation of the group G. It is clear that the dimension of the representation equal to the number of independent generators T_a. Thus the structure constants of arbitrary Lie group create the adjoint representation . Let us remark here that in general generators (85) are not Hermitian operators (see below the conditions which will lead to Hermiticity of T_a).

It is possible to transform the linear space spanned by generators T_a into a vector space: it is necessary to introduce some suitable scalar product. The natural candidate is the real symmetric matrix:

$$tr(T_aT_b). \tag{87}$$

Now we will show that this scalar product can be chosen in a simplest canonical form. The reason of such freedom is in opportunity to perform a linear transformations of generators of Lie algebra:

$$X_a \longrightarrow X'_a = M_{ab}X_b \tag{88}$$

with some matrix M_{ab}. This transformation induces the corresponding change of structure constants through commutation relations of Lie algebra:

$$[X'_a, X'_b] = iM_{ad}M_{be}f_{deg}(M^{-1})_{gc}X'_c. \tag{89}$$

Therefore, the structure constants are transformed as

$$f_{abc} \longrightarrow f'_{abc} = M_{ad}M_{be}(M^{-1})_{gc}f_{deg}. \tag{90}$$

Correspondingly, we have to define new generators of the adjoint representation using transformed structure constants f'_{abc} :

$$(T'_a)_{bc} = M_{ad} M_{be} (M^{-1})_{gc} (T_d)_{eg}, \tag{91}$$

or, equivalently,

$$T'_a = M_{ad} M T_d M^{-1}, \tag{92}$$

i.e. to act by similarity transformation plus additional linear transformation acting onto index a.

If we considered the trace $Tr(T_a T_b)$, this matrix does not feel the similarity transformation in (92), but it is transformed by linear transformation of each indices $a, b, ...$ of generators $T_a, T_b, ...$:

$$Tr(T'_a T'_b) = M_{ac} M_{bd} Tr(T_c T_d). \tag{93}$$

Thus by choosing a suitable (orthogonal) matrix M, one can diagonalize the matrix $Tr(T_a T_b)$:

$$Tr(T_a T_b) = \kappa_a \delta_{ab}, \tag{94}$$

where no summation on index a in the r.h.s. It is also possible to choose all values κ_a with the same modulus (for example, equal to unity). But one can not change the sign of κ_a. Below we will consider only the Lie algebras with all κ_a positive. this choice corresponds to the compact Lie algebras.

Thus we will use the basis for T_a such that

$$Tr(T_a T_b) = \kappa \delta_{ab} \tag{95}$$

with universal positive κ. Just in this basis the structure constants become completely antisymmetric, since

$$f_{abc} = -i(\kappa)^{-1} Tr([T_a, T_b] T_c). \tag{96}$$

This expression for f_{abc} is completely antisymmetric due to cyclic property of trace. The related conclusion is that in this basis operators T_a are Hermitian (since they are pure imaginary and antisymmetric).

One important statement about the adjoint representation is formulated by the following theorem.

Theorem.

The adjoint representation of the simple Lie algebra is irreducible .

5 Group SU(2)

5.1 Algebra su(2).

This is the simplest compact non-Abelian Lie group. Its algebra includes three generators with well known commutation relations, which coincide with the algebra of angular momentum:

$$[J_a, J_b] = i\epsilon_{abc} J_c, \tag{97}$$

where ϵ_{abc} is a completely antisymmetric tensor $\epsilon_{123} = 1$, playing the role of structure constants in this case. There are different ways to construct irreducible representations of the group SU(2). We will use the method of the highest weight, which has essential advantage: it can be naturally generalized to arbitrary compact Lie group.

Let us consider an $N-$dimensional irreducible representation of the algebra su(2). Since, due to (97), no two generators among J_a, $a = 1, 2, 3$ commute with each other, we can diagonalize only one of them. From now on, in the space of representation we will choose the basis where generator J_3 is diagonal. And let us define the element of this basis with **highest eigenvalue** of J_3:

$$J_3|l, \rho\rangle = l|l, \rho\rangle, \tag{98}$$

where index ρ reflects the possible degeneracy of states with the same value of J_3. We choose also the basis which is orthonormal in ρ:

$$\langle l, \rho|l, \rho'\rangle = \delta_{\rho\rho'} \tag{99}$$

It is useful to define, instead of $J_{1,2}$, the operators:

$$J^{\pm} \equiv \frac{J_1 \pm iJ_2}{\sqrt{2}}, \tag{100}$$

which satisfy

$$[J_3, J^{\pm}] = \pm J^{\pm} \tag{101}$$

and

$$[J^+, J^-] = J_3. \tag{102}$$

If the eigenvectors of J_3 are enumerated by number m :

$$J_3|m\rangle = m|m\rangle, \tag{103}$$

then the commutation relations (101) give:

$$J_3 J^{\pm}|m\rangle = (m \pm 1)J^{\pm}|m\rangle. \tag{104}$$

We expect that this relation (and analogous relation for arbitrary compact group) will allow to construct the complete irreducible representations starting from highest state (98).

At first, since $|l\rangle$ has the highest eigenvalue of J_3, we have:

$$J^+|l,\rho\rangle = 0, \tag{105}$$

because the eigenvalue $l+1$ does not exist by definition. Vice versa, the action of J^- lead to the state with eigenvalue $l-1$:

$$J^-|l,\rho\rangle = K_l(\rho)|l-1,\rho\rangle \tag{106}$$

with $K_l(\rho)$ - normalization constant. It is easy to show that from the commutation relation (102) it follows:

$$(K_l(\rho'))^\star K_l(\rho)\langle l-1,\rho'|l-1,\rho\rangle = l\langle l,\rho'|l,\rho\rangle = l\delta_{\rho'\rho}. \tag{107}$$

If we will choose the constants as $K_l(\rho) = \sqrt{l}$, the states $|l-1,\rho\rangle$ will be orthonormal for different values of ρ, ρ'....

In turn, acting again by J^+ we obtain:

$$J^+|l-1,\rho\rangle = \frac{1}{K_l}J^+J^-|l,\rho\rangle = \frac{1}{K_l}[J^+,J^-]|l,\rho\rangle = \frac{l}{K_l}|l,\rho\rangle = K_l|l,\rho\rangle. \tag{108}$$

It is important that the action by operators $J^{\pm}$ does not change the values of ρ.

We have to repeat the procedure above starting now from the eigenvector $|l-1,\rho\rangle$. Then, introducing new constant K_{l-1}, we obtain:

$$J^-|l-1,\rho\rangle = K_{l-1}|l-2,\rho\rangle \tag{109}$$

and

$$J^+|l-2,\rho\rangle = K_{l-1}|l-1,\rho\rangle.$$

Thus we obtain the chain of relations for orthonormal states $|l-k,\rho$:

$$J^-|l-k,\rho\rangle = K_{l-k}|l-k-1,\rho\rangle \tag{111}$$

$$J^+|l-k-1,\rho\rangle = K_{l-k}|l-k,\rho\rangle,$$

where all real constants K_{l-k} satisfy the recurrent relations:

$$K_{l-k}^2 = K_{l-k+1}^2 + l - k. \tag{113}$$

From these relations one may derive that:

$$K_q = \frac{\sqrt{(l+q)(l-q+1)}}{\sqrt{2}}. \tag{114}$$

Now we have to remind the initial condition that the representation is finite dimensional. Therefore, at some moment the action of J^- will give zero:

$$J^-|l-q,\rho\rangle = 0, \tag{115}$$

i.e.

$$K_{l-q} = 0. \tag{116}$$

Since $q+1$ is positive, the only option is:

$$l = q/2 \tag{117}$$

for some integer nonnegative value of q.

From the construction above, one can notice that all generators of the algebra did not change the value of ρ. Thus the irreducible representation has the same value of this index, and it can be dropped in all formulas. The constructed irreducible representation is characterized by the integer or half-integer parameter l, the highest eigenvalue of J_3, and the dimensionality of this representation is $2l+1$. The method which we described is called "the highest weight" method, since the eigenvalues of analogue of J_3 for the general compact group will be called "weights". Therefore, the value of l is the highest (maximal) value of weight.

It is clear that we can start from an arbitrary (in general, reducible) finite dimensional representation . Then we will choose the highest value of weight, and we will construct the whole irreducible representation by action of J^- as above. After that, we will find the next highest weight among the rest states, and we will repeat the procedure. It is evident that the number of such steps will be finite. In this way, we will decompose an arbitrary finite dimensional representation onto direct sum of irreducible representations by the method of highest weight.

Alternative way to build the reducible representation from some set of irreducible representations exists. We mean, to use the direct product procedure (see Subsection 2.2). Indeed, let us choose the basis in the space of direct product as:

$$|i,x\rangle = |i\rangle|x\rangle, \tag{118}$$

where $|i\rangle$ and $|x\rangle$ are the basis for irreducible representations D_1 and D_2 of dimensionalities n and m, respectively. Then we may define the reducible representation $D_1 \bigotimes D_2$ of dimensionality nm by formulas:

$$(D_1 \bigotimes D_2)(\gamma)(|i,x\rangle) = (D_1)(\gamma)(|i\rangle(D_2)(\gamma)|x\rangle). \tag{119}$$

The corresponding generators are defined as:

$$(X_a^{D_1 \otimes D_2})_{ix,jy} = (X_a^{D_1})_{ij}\delta_{xy} + \delta_{ij}(X_a^{D_2})_{xy}. \tag{120}$$

This process can be continued to obtain more complicated representations . The particular case of this procedure corresponds to construction when irreducible representations $D_1, D_2, D_3, ...$ coincide with "spin 1/2" representation $l = 1/2$.

5.2 Isospin

The idea to unite protons and neutrons as a unique particle - nucleon - is very old. The priority belongs probably to Heisenberg, and the main purpose was to simplify investigation of nuclei by taking into account the known similarity of strong interaction forces - charge independence - in all three pairs: pp, $nn\,pn$.

In order to formalize this idea, it is necessary to introduce the nucleon doublet:

$$N = \begin{pmatrix} p \\ n \end{pmatrix} , \tag{121}$$

which realizes the simplest nontrivial representation of the group of internal symmetry. It was natural to suppose that this group is just SU(2), and this doublet is analogous to "spin 1/2" representation of SU(2). Then proton corresponds to the state with eigenvalue $+1/2$ of the operator T_3, analogous to J_3 of Subsections 6.1 and 6.2, and neutron - to the state with the eigenvalue $-1/2$. We have to notice that in Nuclear Physics sometimes the opposite definition is used: proton has eigenvalue $-1/2$, neutron - $+1/2$. The reason is very natural, since as a rule the number of neutrons in nuclei is bigger than the number of protons. Actually, the group of internal symmetry should be called isobaric group (isobars are nuclei with the same atomic number - the number of nucleons), but by some historical reasons commonly used name is isotopic group.

In this Subsection we will use the language of creation and annihilation operators. In particular, the members of nucleon doublet (121) will be written as:

$$|p,\alpha\rangle = a^\dagger_{N,+1/2,\alpha}|0\rangle$$

$$|n,\alpha\rangle = a^{\dagger}_{N,-1/2,\alpha}|0\rangle, \tag{123}$$

where the **nucleon** creation operators $a^{\dagger}_{N,\pm 1/2,\alpha}$ include indices of eigenvalue $\pm 1/2$ of operator T_3 and of other quantum numbers α (spin, coordinates or momenta, etc). By $|0\rangle$ we denote the vacuum state. The index N means that these operators create nucleons, since later on we will consider other strong interacting particles in the frameworks of isotopic group as well. The hermitian conjugated operators

$$a_{N,+1/2,\alpha} \tag{124}$$

play the role of annihilation operators in a sense:

$$a_{N,+1/2,\alpha}|0\rangle = 0. \tag{125}$$

As usual, the introduced creation and annihilation operators of nucleons have to satisfy canonical anticommutation relations:

$$\{a^{\dagger}_{N,k,\alpha}, a_{N,k',\alpha'}\} = \delta_{kk'}\delta_{\alpha\alpha'} \tag{126}$$

$$\{a^{\dagger}_{N,k,\alpha}, a^{\dagger}_{N,k',\alpha'}\} = \{a_{N,k,\alpha}, a_{N,k',\alpha'}\} = 0 \quad k,k' = \pm 1/2. \tag{127}$$

Thus, one can construct the multiparticle nucleon states by means of the creation operators:

$$a^{\dagger}_{N,k_1,\alpha_1} a^{\dagger}_{N,k_2,\alpha_2} ... a^{\dagger}_{N,k_M,\alpha_M}|0\rangle = |M\ nucleons; k_1,\alpha_1,k_2,\alpha_2 ... k_M,\alpha_M\rangle. \tag{128}$$

These states are totally antisymmetric under exchange of quantum numbers of arbitrary pair of particles. Therefore, their wave functions satisfy the generalized Pauli principle, or generalized exclusion principle (generalized, since the isotopic quantum numbers are also interchanged).

The creation and annihilation operators allow to build different physically interesting multiparticle operators. for example, the operators

$$N_p = a^{\dagger}_{N,k=1/2,\alpha} a_{N,k=1/2,\alpha}; \quad N_n = a^{\dagger}_{N,k=-1/2,\alpha} a_{N,k=-1/2,\alpha} \tag{129}$$

count the number of protons and of neutrons, respectively, while the operator

$$N_p + N_n = a^{\dagger}_{N,k=1/2,\alpha} a_{N,k=1/2,\alpha} + a^{\dagger}_{N,k=-1/2,\alpha} a_{N,k=-1/2,\alpha} \tag{130}$$

is the operator of the number of nucleons. It is easy to check by action of operators (129) and (130) onto the states (128) and by using commutation relations (126) and (127).

The action by operators

$$R_{pn} = a^{\dagger}_{N,k=-1/2,\alpha} a_{N,k=1/2,\alpha} \quad R_{np} = a^{\dagger}_{N,k=1/2,\alpha} a_{N,k=-1/2,\alpha} \tag{131}$$

replace all neutrons by protons (R_{np}) or vice versa (R_{pn}) in the same coordinate-spin states. These operators must commute with the Hamiltonian of strong interactions, being its symmetry operators.

Now we will consider the operators introduced above from the point of view of SU(2). Indeed, let us identify the operators R_{pn} and R_{np} from (131) with the raising and lowing operators $T^{\pm}$ of SU(2):

$$T^{+} = \frac{1}{\sqrt{2}} R_{np}; \quad T^{-} = \frac{1}{\sqrt{2}} R_{pn}. \tag{132}$$

The canonical commutation relations (126), (127) allow to calculate the commutator in terms of operators (129) of number of particles N_p, N_n :

$$\begin{aligned} [T^{+}, T^{-}] &= \frac{1}{2}(\sum a^{\dagger}_{N,k=1/2,\alpha} a_{N,k=1/2,\alpha} + \sum a^{\dagger}_{N,k=-1/2,\alpha} a_{N,k=-1/2,\alpha}) = \\ &= \frac{1}{2}(N_p + N_n) \equiv T_3. \end{aligned} \tag{133}$$

Thus, invariance of the strong interaction Hamiltonian under operators $T^{\pm}$ leads immediately to invariance under operator T_3, the fact seems to be very clear physically.

Analogously one can check from the explicit forms of operators, that

$$[T_3, T^{\pm}] = \pm T^{\pm}, \tag{134}$$

leading us to the Lie algebra su(2) described in this Section.

It is useful to represent the generators T_1, T_2, T_3 with the help of Pauli σ-matrices. Indeed, these operators can be expressed as:

$$T_a = \sum a^{\dagger}_{N,k',\alpha} (\frac{1}{2}\sigma_a)_{k'k} a_{N,k,\alpha} = \sum a^{\dagger}_{N,\alpha} (\frac{1}{2}\sigma_a) a_{N,\alpha}, \tag{135}$$

where in the last equation the creation operators are considered as row vectors over the index k' and the annihilation operators - as column vector over k.

One can check that (135) satisfy the necessary commutation relations with creation and annihilation operators:

$$\begin{aligned} &[T_a, a^{\dagger}_{N,k,\alpha}] = [a^{\dagger}_{N,k',\beta} (\frac{1}{2}\sigma_a)_{k'k''} a_{N,k'',\beta}, a^{\dagger}_{N,k,\alpha}] = \\ &= a^{\dagger}_{N,k',\beta} (\frac{1}{2}\sigma_a)_{k'k''} \{a_{N,k'',\beta}, a^{\dagger}_{N,k,\alpha}\} - \{a^{\dagger}_{N,k',\beta}, a^{\dagger}_{N,k,\alpha}\} (\frac{1}{2}\sigma_a)_{k'k''} a_{N,k'',\beta} = \\ &= a^{\dagger}_{N,k',\alpha} (\frac{1}{2}\sigma_a)_{k'k}. \end{aligned} \tag{136}$$

This approach allows to study the properties of multiparticle states which contain not only nucleons but also other strong interacting particles. For example, the pion isotopic triplet (isospin 1) adds new terms into the nucleon isotopic generators T_a given by (131), (132) and (133). These additional pionic terms are expressed in terms of creation and annihilation operators $a^\dagger_{\pi,k,\alpha}$ and $a_{\pi,k,\alpha}$ with $k = 0, \pm 1$, as follows:

$$T^+ = a^\dagger_{\pi,k=+1,\alpha} a_{\pi,k=0,\alpha} + a^\dagger_{\pi,k=0,\alpha} a_{\pi,k=-1,\alpha}; \tag{137}$$

$$T^- = a^\dagger_{\pi,k=-1,\alpha} a_{\pi,k=0,\alpha} + a^\dagger_{\pi,k=0,\alpha} a_{\pi,k=+1,\alpha}; \tag{138}$$

$$T_3 = a^\dagger_{\pi,k=+1,\alpha} a_{\pi,k=+1,\alpha} - a^\dagger_{\pi,k=-1,\alpha} a_{\pi,k=-1,\alpha}; \tag{139}$$

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