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GROUP-THEORETICAL METHODS IN QUANTUM FIELD THEORY

Учебно-методическое пособие

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В учебно-методическом пособии изложены наиболее эффективные методы теории представлений групп и алгебр Ли, применяемые в физике элементарных частиц и квантовой теории поля.
Пособие предназначено для студентов физических и физико-математических специальностей 4-7-го курсов, аспирантов, соискателей и специалистов, интересующихся вопросами теоретической и математической физики.

Contents

Introduction

Lie algebras have found fruitful applications in diverse areas, such as analytical mechanics, atomic and nuclear spectra and aberration phenomena in optics. The most effective was the usage of Lie algebraic methods in the theory of elementary particles, gauge theories, gravity and string theory. Lie algebras are closely connected to many other algebraic structures in mathematical and theoretical physics, such as quantum groups, vertex operator algebras and fusion rules, the structures that provide a new insight in physical theories. Lie algebras give an optimal initiation to this rich reservoir of mathematical concepts.

In the elementary particles theory most of continuous symmetries are broken. The broken or deformed symmetries underlie understanding of the structure of weak interactions, superconductivity, critical phenomena and applications of the elementary particles theory to the solution of cosmological problems. The major approach in the description of symmetries in quantum physics is the representation theory. Any violation of a symmetry induces processes of branching of representations. Therefore it is necessary to have an explicit construction of embeddings and reduction of representations – the branching rules. The scheme of branching allows to install a multiplet structure for an integral symmetry and its behaviour in different types of deformations, enables obviously to build harmonic expansions for fields and to analyze spinor structures. It concerns in particular models of conformal field theory[1], spontaneous compactification[2], dimensional reduction, and also models of harmonic superspace[3].

In our lectures we shall consider an effective method of reduction of representations for regular embeddings. It is supposed, that the student is familiar with main concepts of the representation theory in the scope of our lectures "Introduction to the continuous group theory " for the students of theoretical specializations of SPbSU Department of Physics.

In the first Section the main notions necessary for our study are introduced and commented. In Section 2 the correspondence between the recurrence properties for embeddings is studied in terms of relative multiplicities. The visual interpretation of a "star" of shifts as a special case of the "fan" uncovers internal logic of the relative multiplicities method and justifies its universality. This approach allows to obtain the multiplicities of irreducible components of reduced representation via the relation similar to the Costant

formula [9]. It is shown, that this formula contains the compensatory multiplicities and, as a result, simplifies the necessary calculations. In the third section the applications of the above mentioned method are presented.

Chapter 2 is devoted to the relativistic symmetry and the tools that are provided by the group theory to describe relativistic elementary particles. In the fourth section the main aspects of induced representations are studied. In the theory of induced representations the most powerful and useful approach (especially in applications to the quantum field theory) is the little group method described in section 5. Here the irreducible representations are also constructed for the general Poincare symmetry including reflections. Finally in the sixth section the unitary and anti-unitary representations of the Poincare group with reflections are used to demonstrate how the relativistic wave equations arise from the irreducibility conditions.

Chapter 1.

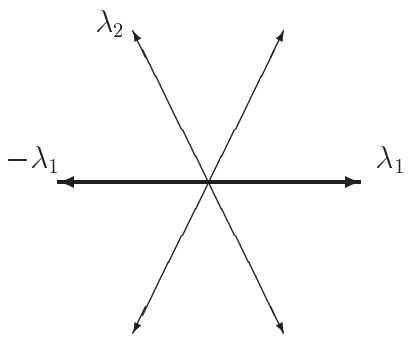
Injections and branching rules

1 *Formal algebras of weights*

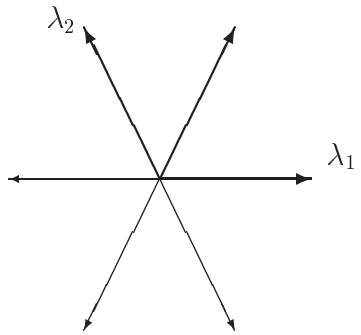
1.1 *Definitions and notations*

g - simple Lie algebra
$\widetilde{g}$ - reductive regular subalgebra in g
$B(g)$ - Cartan subalgebra in g

$\Lambda, \widetilde{\Lambda}$ - root systems of algebras g and $\widetilde{g}$. As far as the injection is regular the root system $\widetilde{\Lambda}$ of $\widetilde{g}$ is a subsystem of Λ. The classical example is a pair of roots $\lambda_1, -\lambda_1$ in any system Λ, it corresponds to an $sl(2, C)$-subalgebra in g. Here is how it looks in the case $sl(2, C) \subset sl(3, C)$:

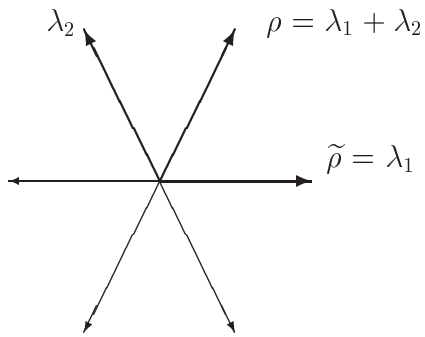


Δ, $\widetilde{\Delta}$ - appropriate sets of the positive roots; for regular embeddings $\widetilde{\Delta}$ it is a subsystem of positive roots of a semisimple subalgebra in g. In our example the set Δ for $sl(3)$ is indicated by thick vectors:



S, $\widetilde{S}$ - sets of basic (or simple) roots. In our example – the injection $sl(2) \to sl(3)$ – these are the sets $\{\lambda_1, \lambda_2\}$ and $\{\lambda_1\}$.

$\rho, \widetilde{\rho}$ - half-sum of positive roots for g and $\widetilde{g}$ correspondingly; this element of the root space is also called the Weyl vector.



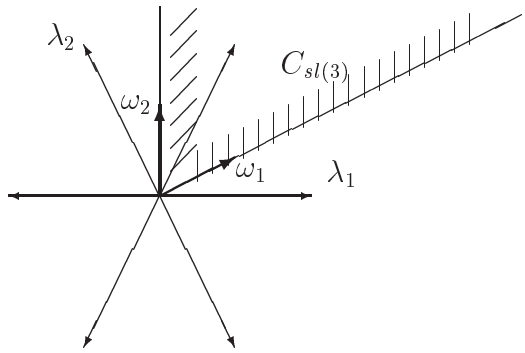
W, V - Weyl groups for g and $\widetilde{g}$. It is a discrete groups generated by reflection operators s_λ for algebra g and $s_{\widetilde{\lambda}}$ - for $\widetilde{g}$. Let's remind, that in the root space to each root $\lambda \in \Lambda$ corresponds the reflection with respect to the hyperplane, orthogonal to vector λ.

$$s_\lambda \xi = \xi - 2\frac{(\lambda, \xi)}{(\lambda, \lambda)}\lambda.$$

It immediately follows, that for the generating operators $\{s_\lambda \mid \lambda \in \Lambda\}$ the property is fair $\det s_\lambda = -1$.

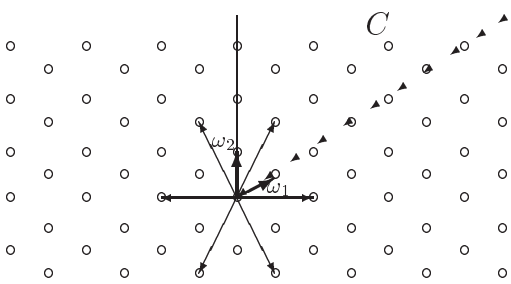
C, $\widetilde{C}$ — the Weil chamber, principal with respect to S and $\widetilde{S}$. The Weil chamber consists of all such vectors of weight space, that have nonnegative and not equal to zero simultaneously coordinates in the base of fundamental

weights. Fundamental weights (ω_1 and ω_2 for sl(3)) lie on the boundary of the Weil chamber.



$\overline{C}, \overline{\widetilde{C}}$ — closures of the appropriate Weil chambers. The transformations of a Weyl group on $\overline{C}$ generates the whole weight space. In many cases for the analysis of the weight diagram as a whole it is enough to consider the situation in the principal Weil chamber.

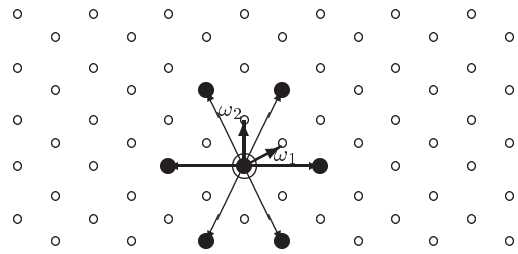
P_g, $P_{\widetilde{g}}$ - weight lattice for g and $\widetilde{g}$. The lattice of a simple Lie algebra can be supplied with a natural group composition – the Abelian group – the additive group of vectors. The weight lattice is generated by the set of fundamental weights. For $sl(3)$ it looks as follows:



$\mathcal{E}$, $\widetilde{\mathcal{E}}$ – formal associative algebras with unit generated by the groups P_g and $P_{\widetilde{g}}$. The algebras $\mathcal{E}$ are group algebras [6] of groups P_g. Their elements are the linear combinations a formal exponents e^{ν} with factors from the base field. To any weight ν of the weight diagram of an irreducible representation d^{μ} that have the multiplicity k the formal element ke^{ν} is attributed. (For more details about properties of algebras of formal series

see [11].) To construct algebras $\mathcal{E}$ the formal exponents e^{ω} for fundamental weights $\{\omega\}$ can be used. The multiplication in $\mathcal{E}$ is induced by the weight composition in the group P_g: $e^{\beta} \cdot e^{\gamma} = e^{\beta+\gamma}$.

N_{μ} – weight diagram of representation d^{μ} with highest weight μ. Here is the diagram of the adjoint representation $d^{\mu=\rho}$ of the algebra $sl(3)$ (the weights of N_{ρ} are red):



m_{ν} – full multiplicity of the weight ν. For example, in the representation $d^{\rho}_{sl(3)}$ (who's diagram is presented above) the central weight has $m_0 = 2$.

m'_{ν} – induced multiplicity of weight (in reducible representation); the multiplicity of weight ν, as an element of the diagrams N_{ξ} for all the subrepresentations with weights ξ strictly higher than ν.

n_{ν} - relative multiplicity of weight ν,

$$n_{\nu} = m_{\nu} - m'_{\nu}$$

In our example $sl(2,C) \subset sl(3,C)$ and $d^{\rho}_{sl(3)}$ the weights of N_{ρ} belonging to the subspace generated by λ_1 form the diagram of the reducible representation of the subalgebra $sl(2)$. In this subspace two weights have nonzero induced multiplicities: $m'_0 = m'_{-\lambda_1} = 1$ and two weights have nonzero relative multiplicities: $n_0 = n_{\lambda_1} = 1$.

$\widetilde{n}_{\nu}(g, \widetilde{g}, \mu, \nu)$ – anomalous relative multiplicity, the function in weight space of algebra g,

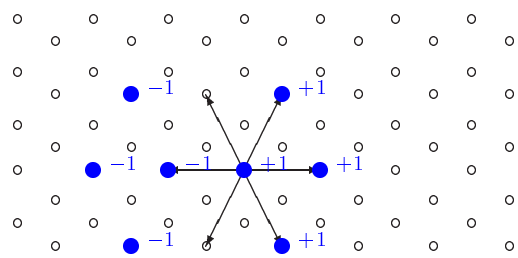
$$\begin{aligned} &\widetilde{n}(g, \widetilde{g}, \mu, \nu) \equiv \widetilde{n}_{\nu} = \\ &= \begin{cases} det(v) n_{\widetilde{\mu}} & \text{for } \{\widetilde{\mu} \in M \mid v(\widetilde{\mu} + \widetilde{\rho}) - \widetilde{\rho} = \nu\} \\ 0 & \text{in other cases} \end{cases} , \end{aligned} \tag{1}$$

here M is the set of highest weights appearing in the decomposition of

representation $d^{\mu}(g)_{\downarrow\tilde{g}}$

$$M = \{\widetilde{\mu} \mid n_{\widetilde{\mu}} \neq 0\}.$$

On the figure below the blue spots indicate the points where the function $\widetilde{n}(g, \widetilde{g}, \mu, \nu)$ has nonzero (positive and negative) values for the representation $d^{\rho}_{sl(3)}$.



$ch\, d^{\mu}$ – formal character of representation d^{μ}; an element of algebra $\mathcal{E}$, corresponding to the weight diagram N_{μ} of representation d^{μ}:

$$ch\, d^{\mu} = \sum_{\nu} m_{\nu} e^{\nu}$$

Such algebraization of the weight diagrams and operations with them appears to be a useful tool for the analysis of irreducible representations. The following element is the formal character of the representation $d^{\rho}_{sl(3)}$,

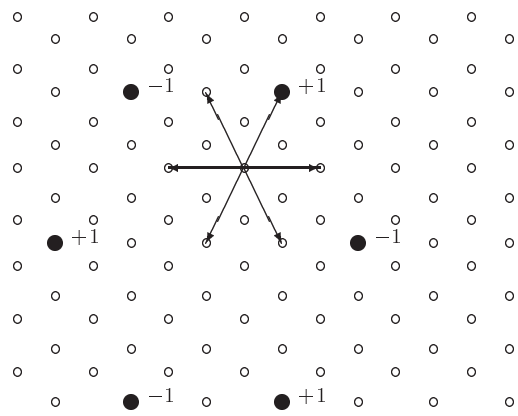
$$ch\, d^{\rho}_{sl(3)} = e^{\rho} + e^{\lambda_1} + e^{\lambda_2} + e^{-\lambda_1} + e^{-\lambda_2} + e^{-\rho} + 2.$$

Ψ^{μ}, $\widetilde{\Psi}^{\widetilde{\mu}}$ – elements of algebras $\mathcal{E}$, $\widetilde{\mathcal{E}}$, corresponding to sets of anomalous weights for representations d^{μ}, $\widetilde{d}^{\widetilde{\mu}}$:

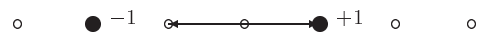
$$\Psi^{\mu} = \sum_{w \in W} \det(w) e^{w(\mu+\rho)-\rho}, \tag{2}$$

$$\widetilde{\Psi}^{\widetilde{\mu}} = \sum_{v \in V} \det(v))^{v(\widetilde{\mu}+\widetilde{\rho})-\widetilde{\rho}}. \tag{3}$$

For our example,the representation $d^{\rho}_{sl(3)}$, the set of weights defining the element Ψ^{ρ} is indicated by the following 6 red points:



Compare the previous set with the anomalous points of the subrepresentation $d^{\mu=2\omega}_{sl(2)}$:



In the following study we shall see that the elements Ψ^μ describe the properties of the representations with respect to the shifts from $\Lambda(g)$ that is with respect to the action of the ladder operators of the representation d^μ.

2 *Local recurrence for relative multiplicities*

2.1 *Reduction schemes for relative multiplicities*

As you know one of the most popular tools to perform a decomposition of an irreducible representation with respect to subalgebra is the Gelfand-Tzetlin method [5] (its detailed exposition can be found in the book by A.Barut and R.Ronchka [6]), however it is applicable only for classical Lie algebras of one type (unitary, orthogonal) and allows to perform the decomposition, when the dimension of the defining representation changes by one ($su(n) \downarrow su(n-1)$, $so(2n) \downarrow so(2n-1)$ etc.). Using the properties *of nega-*

tive Young tableau and the Littlewood theorem for embeddings of unitary algebras of a type

$$su(p) \oplus su(q) \oplus u(1) \to su(q+p),$$

it is possible to formulate a simple algorithm giving not only the multiplicities of irreducible subrepresentations, but also their highest weights [7]. But we must remember that the attempts to introduce the generalized Young tableau for orthogonal and symplectic algebras gave no results that could be used in practical calculations [15].

As for methods of direct analysis of weight diagrams, such as projective operators methods [4], they are really applicable only for algebras with small ranks and small dimensions of representations. Their application implies the knowledge of absolute multiplicities of all weights of the diagram. The calculations necessary to evaluate these multiplicities using the formulas, such as Freydental, Costant or Racha relations [9], are extremely cumbersome even for their modified versions. The investigations of the problem of reduction based on the theory of characters and Shcur functions [15, 16, 17] does not contain results, suitable for practical calculations. In view of this situation the empirical rules of decomposition were repeatedly offered [18]. However all of them have limited and badly controllable application area. As it has been marked in work [**?**], there is only one "universal method of reduction" — to construct completely the weight diagram, to project it to the root subspace of a subalgebra and to decompose the obtained representation. For the general case of regular injections the most extensive research program has been executed in a series of works of R.V.Mudi, J. Patera, R.T.Sharp and F.Gingras [19]. Their constructions are based on properties of Weil group orbits in the weight spaces and on the generating function technique. However within the framework of this approach it was not possible to get the final algorithm for calculations of multiplicities. Summing up we can conclude that most of the usual methods of reduction of representations (listed above) are insufficiently universal and insufficiently effective for our needs [4].

In this section we shall show, that the solution of the problem of reduction is still possible and that to find such a solution the recurrent relations for multiplicities can be successfully used. A number of results will be proved that will allow to formulate a universal method of reduction. The suggested approach is based on properties of root systems and Weil groups of algebras

and it is universal in the sense that allows to build decomposition for any representation of any classical algebra when the injection is regular.

Consider the representation d^{μ}_{g} with the highest weight μ be and the diagram N_{μ}. In this section the problem of finding multiplicities $n_{\widetilde{\mu}}$ of subrepresentations $d^{\widetilde{\mu}}(\tilde{g})$ is solved for the general form of decomposition

$$d^{\mu}(g)_{\downarrow\tilde{g}} \quad = \quad \oplus_{\widetilde{\mu}} \;\; n_{\widetilde{\mu}} d^{\widetilde{\mu}}(\tilde{g}). \tag{4}$$

All the constructions of this section will be demonstrated for the case where the injection $\widetilde{g} \to g$ is maximal and the subalgebra $\widetilde{g}$ is regular and reductive (i.e. can be presented as the direct sum simple and Abelian subalgebras; for more details about reductive algebras see [14]).

With respect to the shifts from $\Lambda_{\tilde{g}}$ the weight diagram N_{λ} is decomposed into mutually non-intersecting orbits $\Omega^{\mu,\tilde{g}}_{i}$, which will be called floors of the decomposition of the weight diagram. Let i be the number of a floor (when this will not cause misunderstandings, orbits will be designated Ω_i). In the general case the set of weights $\nu \in \Omega_i$ determines the diagram N_i of a reducible representation d_i for the subalgebra $\tilde{g}$. Let m_{ν} be the absolute multiplicity of weight ν in representation d^{μ} and $\pi(\nu)$ — the projection of the weight ν on the root subspace $\Lambda_{\tilde{g}}$ (naturally, $P(\Omega_i) = N_i$). Let's define a set of highest weights $\{\xi\}$ for the irreducible components of representations d_i:

$$d_i = \oplus \; n^{i}_{\xi} \; d^{\xi} \;\; \{\xi\} \;\; \equiv \;\; \{\xi \mid \xi \in \{\nu\} \cap \pi(\nu), \quad \nu \in \Omega_i\}.$$

For any $\nu \in \Omega_i$ we shall introduce a subset

$$\{\xi''\} \;\; \equiv \;\; \{\xi'' \in \{\xi\} \mid \xi'' > \pi(\nu)\}$$

and subrepresentations

$$d''_i \;\; = \;\; \oplus \; n^{i}_{\xi''} \; d^{\xi''}. \tag{5}$$

Let m'_{ν} be the multiplicity of the weight $\pi(\nu)$ in representation (5). For any weight ν of the representation d^{μ} the full multiplicity m_{ν} can be presented as the sum

$$m_{\nu} = m'_{\nu} + n_{\nu}. \tag{6}$$

with the relative multiplicity n_{ν}. When ν belongs to the principal Weil chamber, its relative multiplicity coincides with the corresponding factor in the decomposition (4). Thus, the reduction process is can be considered as the evaluation of relative multiplicities of weights in N^{λ}.

As it will be shown below, the general recurrent relation for regular injections can be naturally formulated in terms of anomalous relative multiplicities $\widetilde{n}_\nu$.

2.2 *Recurrence relations*

The reduction of an irreducible representation is at the same time the reduction of its weight diagram N_λ and thus of its formal character:

$$ch\, d^\lambda = \sum_{\widetilde{\mu}} n_{\widetilde{\mu}} ch\, d^{\widetilde{\mu}}. \tag{7}$$

Let's consider an element $D(g)$ of algebra $\mathcal{E}$

$$D(g) = e^\rho \prod_{\alpha\in\Delta} (1 - e^{-\alpha}).$$

It can be proved [9] that it is proportional to the set of anomalous multiplicities of the trivial representation $d^{\mu=0}$:

$$D(g) = \sum_{\sigma\in W} \det(\sigma) e^{\sigma\rho}. \tag{8}$$

The product of this element and the weight diagram $ch\, d^\xi$ coincides (to within the factor e^ρ) with the set of the anomalous multiplicities Ψ^ξ

$$ch\, d^\xi \cdot D(g) = \Psi^\xi \cdot e^\rho.$$

It is easy to check this equality on diagrams of fundamental representations. (Its detailed proof can be found in the paper by P. Cartier [20].) The same relation rewritten in the form

$$ch\, d^\xi = \frac{\Psi^\xi}{\prod_{\alpha\in\Delta}(1 - e^{-\alpha})}, \tag{9}$$

wears the name of the Weil formula for formal characters.

Returning to the relation (7) and taking into account the injection $\widetilde{\Delta} \to \Delta$, we use the Weil formula to establish a connection between the anomalous elements Ψ^μ and $\widetilde{\Psi}^{\widetilde{\mu}}$

$$(\prod_{\Delta\setminus\widetilde{\Delta}} (1 - e^{-\alpha}))^{-1} \Psi^\mu = \sum_\mu n_\mu \widetilde{\Psi}^{\widetilde{\mu}}. \tag{10}$$

Using the base $\{e^{\xi}\}_{\xi\in\pi_g}$ of the algebra $\mathcal{E}$, it is possible to convert both parts of the relation (10). Expanding the first factor (in the left-hand side) with respect to the base $\{e^{-\xi}\}$

$$(\prod_{\Delta\setminus\widetilde{\Delta}}(1-e^{-\alpha}))^{-1}=\sum_{\xi}K_{\widetilde{g}\subset g}(\xi)e^{-\xi} \tag{11}$$

we obtain the so-called Kostant-Heckman distribution function [21]. The expression (11) together with the formulas (2) and (3) gives the desirable expansion of the expression (10). Now we shall consider a weight $\widetilde{\mu}$ from principal Weil chamber $\widetilde{C}$, i.e. the anomalous weight at $v=e$. Comparing coefficients, we obtain the expression for the relative multiplicity $n_{\widetilde{\mu}}$ in terms of the distribution function $K_{\widetilde{g}\subset g}$

$$n_{\widetilde{\mu}}=\sum_{W}\det(w)K_{\widetilde{g}\subset g}(w(\mu+\rho)-(\rho+\widetilde{\mu})). \tag{12}$$

Let's notice, that the relation (10) is valid on all the weights lattice P_g. Thus, it is possible to rewrite it in the following form:

$$\sum_{\widetilde{\mu}}n_{\widetilde{\mu}}\widetilde{\Psi}^{\widetilde{\mu}}=\sum_{\widetilde{\mu}}n_{\widetilde{\mu}}\sum_{v\in V}\det(v)e^{v(\widetilde{\mu}+\widetilde{\rho})-\widetilde{\rho}}=\sum_{\xi}\widetilde{n}_{\xi}e^{\xi}. \tag{13}$$

As far as all the weights $\{v(\widetilde{\mu}+\widetilde{\rho})-\widetilde{\rho}\}$ are different the coefficients $\widetilde{n}_{\xi}$ here are just the anomalous relative multiplicities (see the formula (1)). The relation (13) together with the expansion (11) and the definition (2) show that the expression (12) remains valid in all the points P_g:

$$\widetilde{n}_{\xi}=\sum_{W}\det(w)K_{\widetilde{g}\subset g}(w(\mu+\rho)-(\rho+\xi)). \tag{14}$$

We shall use this formula to construct the recursion relation for $\widetilde{n}_{\xi}$. First we shall consider the expression (14) for $\mu=0$. In this case $\widetilde{n}_{\xi}$ describes the functional dependence between the anomalous weights multiplicities and the weights for the diagram of the trivial subrepresentation $(\widetilde{d}^0)$:

$$\widetilde{n}_{\xi}=\sum_{V}\det(v)\delta_{\xi,\,v\widetilde{\rho}-\widetilde{\rho}}=\sum_{W}\det(w)K_{\widetilde{g}\subset g}(w\rho-(\rho+\xi)). \tag{15}$$

It is possible to allocate the trivial term (with $w=e$) and to obtain the recursion relation for the distribution function $K_{\widetilde{g}\subset g}$

$$K_{\widetilde{g}\subset g}(\xi)=-\sum_{W\setminus e}\det(w)K_{\widetilde{g}\subset g}(\xi+(w-1)\rho)+\sum_{V}\det(v)\delta_{\xi,\widetilde{\rho}-v\widetilde{\rho}}. \tag{16}$$

Returning to the expression (14) one easily comes to the conclusion that the relation (16) induces the recursion relation for the anomalous relative multiplicities

$$\begin{aligned}\widetilde{n}_\xi = & - \sum_{W\backslash e} \det(w)\widetilde{n}_{\xi+(1-w)\rho} + \\ & + \sum_{W,V} \det(w)\det(v)\delta_{\xi+(1-v)\widetilde{\rho},\, w(\mu+\rho)-\rho}. \end{aligned} \tag{17}$$

2.3 *Reduction to a Cartan subalgebra*

The Cartan subalgebra $b(g)$ can be interpreted as a reductive regular subalgebra in semisimple algebra g.

Our approach will be based on the universal recurrent relation for regular injections which are naturally formulated in terms of anomalous relative multiplicities $\widetilde{n}_\nu$.

Here we shall analyse the structure of the relation (17) and its properties in a simple modelling situation, when a subalgebra $\widetilde{g}$ coincides with a Cartan subalgebra, that is a regular subalgebra with minimal interior structure. Such analysis will allow to reveal fundamental recurrence properties in a weights lattice and also will allow to open interior logic in the universal relation (17) and to be convinced of naturalness of the anomalous multiplicity notion.

Let's put $\widetilde{g} = b(g)$.

Let's remind, that any weight of the diagram N_μ of a representations d^μ corresponds to a subspace of representation space U, invariant with respect to the operation $d^\mu(b(g))$. In other words, the expansion $U = \oplus_\nu U^\nu$ on weight subspaces corresponds to a reduction of the initial representation on subrepresentations of the Cartan subalgebra:

$$d^\mu(g)_{\downarrow b(g)} = \oplus_\nu m_\nu d^\nu(b(g)).$$

Here the relative multiplicity (that is the subrepresentation multiplicity) plays the role of the full multiplicity m of the weight . Coincidence of the notions of full and relative multiplicities is typical for Abelian reductive subalgebras. In fact a representation of an Abelian algebra can induce multiplicity only for the highest weight.

As the Weyl group of a Cartan subalgebra is trivial, $V = e$, the anomalous relative multiplicity also coincides here with the full weight multiplicity.

To check this, it is enough to put $v = e$ in the definition (1), that immediately reduces to the equality

$$\widetilde{n_\nu} = n_\nu = m_\nu.$$

As a result the equality (17) can be read as recurrent relation for full multiplicities of weights of an irreducible representation d^μ:

$$m_\nu = -\sum_{W\setminus e} \det(w) m_{\nu+(1-w)\rho} + \sum_{w\in W} \det(w)\delta_{\nu,\, w(\mu+\rho)-\rho}. \tag{18}$$

The second sum in this expression should be interpreted as anomalous multiplicity of representation $d^\mu(g)$. Analogously to the anomalous relative multiplicities $\widetilde{n}$ (see (1)) the function of anomalous multiplicity $\widetilde{m_\nu}$ for the representations d^μ can be written,

$$\widetilde{m}(g,\mu,\nu) \equiv \widetilde{m}_\nu(\mu) = \\ = \begin{cases} det(w) n_\mu & \text{for } \{\mu \in \pi_g \mid w(\mu+\rho)-\rho = \nu\} \\ 0 & \text{in other cases} \end{cases}. \tag{19}$$

Finally we have

$$m_\nu = -\sum_{W\setminus e} \det(w) m_{\nu+(1-w)\rho} + \widetilde{m}_\nu(\mu). \tag{20}$$

The obtained formula resembles the local recurrence formula [13], but does not coincide with it. The difference is just that in a standard spelling of the formula (20) the anomalous multiplicity of the representation is absent. Correspondingly the clause is made that this relation is to be applied only for $\nu \in N_\mu \setminus \mu$.

It is necessary to underline, that only the item $\widetilde{m_\nu}(\mu)$ contains information about the irreducible representation. Let's consider a growing sequence of the highest weights $\mu_i, i \in Z; \mid \mu_{i+1} \mid > \mid \mu_i \mid$. In the limit $i \to \infty$ for any weight ν of final length we shall receive the usual local recurrence formula [13]:

$$m_\nu = -\sum_{W\setminus e} \det(w) m_{\nu+(1-w)\rho}. \tag{21}$$

Local recurrence, expressed by this expression is an important property of algebra g. It describes the behaviour of weight multiplicities characteristic to *any* irreducible representation of g when $\mid \mu - \nu \mid \leq \mid \mu \mid$. On the contrary,

formula (20) is the relation describing the *local recurrence for the representation* d^μ. Here the representation is fixed naturally – by means of its highest weight. The remaining weights can be found using the formula (20) together with their multiplicities.

Anomalous weights in $\widetilde{m}_\nu(\mu)$, forcing a recursion to break outside of the external contour of the diagram N^μ, finally guarantee the weights to remain inside the diagram of an irreducible representation and also allot its invariance with respect to the Weyl group.

2.4 *Branching for a nonabelian reductive subalgebra*

The relation (17) can be used for precise calculations of multiplicities $\widetilde{n}_\nu$ and n_μ and in some cases it is very effective. But the necessity to estimate continually the complete set of test weights $\{\xi + (1-w)\rho\}$ can make the whole process inconvenient. Let's consider once again the formula (10). The previous transformations were based on the properties of the operator inverse to $\prod_{\Delta\setminus\widetilde{\Delta}}(1-e^{-\alpha})$. Now we shall use it directly and taking into consideration the expression (13), we shall rewrite the relation (10) in the following form:

$$\Psi^\mu = \sum_{w\in W} \det(w) e^{w(\lambda+\rho)-\rho} = \prod_{\Delta\setminus\widetilde{\Delta}} (1-e^{-\alpha}) \cdot \sum_{\xi} \widetilde{n}_\xi e^\xi. \tag{22}$$

The first factor in the formula (22) determines the finite set of weights $\Gamma(\widetilde{g} \subset g)$. Its structure depends only on an injection $\widetilde{g} \subset g$:

$$\prod_{\Delta\setminus\widetilde{\Delta}} (1-e^{-\alpha}) = 1 - \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma) e^{-\gamma}. \tag{23}$$

Here $\mathrm{sign}(\gamma)$ is the parity of a weight γ. Let p be the number of components in the expansion of γ with $\alpha \in \Delta \setminus \widetilde{\Delta}$, then $\mathrm{sign}(\gamma) = (-1)^{p+1}$. In these terms the formula (22) reduces to the following recursion relation:

$$\widetilde{n}_\nu = \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma) \widetilde{n}_{\nu+\gamma} + \sum_{w\in W} \det(w) \delta_{\nu,\, w(\lambda+\rho)-\rho}. \tag{24}$$

Its effectiveness depends mainly on the possibility to construct the set $\Gamma(\widetilde{g} \subset g)$.

As it was stressed in the previous paragraph the element D/e^{ρ} coincides with the set of the anomalous multiplicities of the trivial representation

$$\prod_{\alpha\in\Delta}(1-e^{-\alpha})=\Psi^0. \tag{25}$$

We use this property to convert the expression (23)

$$\prod_{\Delta\setminus\widetilde{\Delta}}(1-e^{-\alpha})=1-\sum_{\gamma\in\Gamma}\mathrm{sign}(\gamma)e^{-\gamma}=(\prod_{\alpha\in\widetilde{\Delta}}(1-e^{-\alpha}))^{-1}\cdot\Psi^0. \tag{26}$$

The anomalous element Ψ^0 is W-invariant and can be factorized with respect to $V\subset W$

$$\Psi^0=\sum_{x\in X}\sum_{v\in V}\det(v\cdot x)e^{(v\cdot x-1)\rho}. \tag{27}$$

Here X is the factor space W/V. It allows to present Γ as a set of weight diagrams of representations $\widetilde{d}$:

$$\begin{aligned}1 \;-\; & \sum_{\gamma\in\Gamma}\mathrm{sign}(\gamma)e^{-\gamma}= \\ = \; & \sum_{x\in X}\det(x)(\prod_{\alpha\in\widetilde{\Delta}}(1-e^{-\alpha}))^{-1}\sum_{v\in V}\det(v)e^{(vx\rho-\widetilde{\rho}+(\widetilde{\rho}-\rho))}= \\ = \; & e^{\widetilde{\rho}-\rho}\sum_{x\in X}\det(x)ch\,\widetilde{d}^{(x\rho-\widetilde{\rho})}.\end{aligned} \tag{28}$$

Representatives of the classes x in (28) always can and should be picked so that the weights $x\rho-\widetilde{\rho}$ are in the principal Weil chamber $\overline{\widetilde{C}}$. Let's notice, that an element Ψ^0 multiplied by $(\prod_{\alpha\in\Delta}(1-e^{-\alpha}))^{-1}$ gives a weight of the trivial representation $d^0(g)$, while being multiplied by $(\prod_{\alpha\in\widetilde{\Delta}}(1-e^{-\alpha}))^{-1}$ generates a set $\Xi(\widetilde{g}\subset g)$ of weight diagrams of representations $\widetilde{g}$.

Let N^{ξ} be the weight diagram of the irreducible representation $\widetilde{d}^{\xi}$ with the highest weight $\xi\in\Xi(\widetilde{g}\subset g)$. Due to the relation (28) the set $\Gamma(\widetilde{g}\subset g)$ is obtained as a union of diagrams N^{ξ}

$$\Gamma=(\bigcup_{\xi\in\Xi(\widetilde{g}\subset g)}N^{\xi})\setminus\{0\}. \tag{29}$$

So to find $\Gamma(\widetilde{g}\subset g)$ it is sufficient to construct the set $\Xi(\widetilde{g}\subset g)$. Apparently the latter depends only on the structure of the factor space W/V and the weights ρ and $\widetilde{\rho}$.

To finish the general exposition of the recursion properties of $\widetilde{n}_\mu$ we demonstrate the interrelations between the two recursion formulas (17) and (24).

Lemma 3- 1 *The recursion relation (17) can be factorized with respect to a subgroup V of W, so that the summation over the factor space $W \setminus V$ is substituted by a summation over Γ.*

Proof. We use the formula (24) to write down the recursion relation for the integral expression $\sum_{v\in V} \det(v)\widetilde{n}_{\nu+(1-v)\widetilde{\rho}}$ and to allocate the first term relevant to $v = e$:

$$\begin{aligned}
\widetilde{n}_\nu &= -\sum_{v\in V, v\neq e} \det(v)\widetilde{n}_{\nu+(1-v)\widetilde{\rho}} - \\
&\quad -\sum_{v\in V}\sum_{\gamma\in\Gamma} \det(v)\ \mathrm{sign}(\gamma)\widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} + \\
&\quad + \sum_{w\in W, v\in V} \det(v)\det(w)\delta_{\nu+(1-v)\widetilde{\rho},\ w(\lambda+\rho)-\rho} = \\
&= \sum_{V,\Gamma\cup\{0\};(v,\gamma)\neq(e,0)} \det(v)\ \mathrm{sign}(\gamma)\widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} + \\
&\quad + \sum_{w\in W, v\in V} \det(v)\det(w)\delta_{\nu+(1-v)\widetilde{\rho},\ w(\lambda+\rho)-\rho}. \qquad (30)
\end{aligned}$$

Comparing this expression with the relation (17) we see that using the set Γ ensures a factorization in the first term of the formula (17). Thus the relation (24) can be called *the factorized recursion formula* for the anomalous relative multiplicities.

2.5 *Fan and star*

In calculations of anomalous relative multiplicities the most powerful is the factorized formula (24). There the summation in the first part is carried out not on the shifted weights $\{\nu + (1 - w)\rho\}$, but only on a set of weights $\Gamma(\widetilde{g} \subset g)$, named a fan. Let us consider once again the set Γ:

$$\prod_{\Delta\setminus\widetilde{\Delta}} (1 - e^{-\alpha}) = 1 - \sum_{\gamma\in\Gamma} \mathrm{sign}(\gamma)e^{-\gamma}. \qquad (31)$$

In the special case $\widetilde{g} = b(g)$ the formula (31) assumes a fairly simple form:

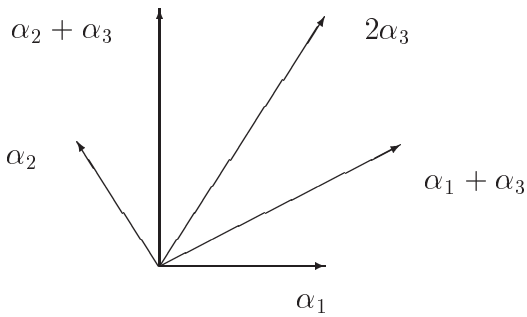
$$\begin{aligned}
\Psi^0 &= \prod_{\alpha\in\Delta}(1-e^{-\alpha}) = \\
&= \sum_W \det(w)e^{(w-1)\rho} \\
&= 1+\sum_{W\setminus e}\det(w)e^{(w-1)\rho} \\
&= 1-\sum_{\gamma\in\Gamma(h(g)\subset g)}\mathrm{sign}(\gamma)e^{-\gamma}.
\end{aligned}$$

Thus, for an injection $b(g) \to g$ the fan Γ is nothing but the set of weights $\{(1-W)\rho\}$. While applying the recurrence in the form (21) it is usually implied the summation of the multiplicities of the weights shifted by the "star" is performed $\{(1-W)\rho\}$ [14]. Above mentioned reasonings show that the "star" is a special case of the fan, when the role of its "plates" play the irreducible representations of the Cartan subalgebra.

Here are the basic conclusions derived from the analysis of the Cartan subalgebra properties (regarded as a reductive regular subalgebra in g):

- The fan $\Gamma(\widetilde{g} \subset g)$, describing an injection $\widetilde{g} \to g$, allows to calculate recursively the multiplicities of subrepresentations for weights $\{\nu \in N_\mu \mid \mid \nu \mid < \mid \mu \mid\}$;
- Anomalous weights Ψ^μ of representations d^μ guarantee the termination of the recursion on Γ and by that form the set of anomalous relative multiplicities for the irreducible representation d^μ.
- Both previous properties are inherent also for sets of full multiplicities of irreducible representations and in that case the star $\{(1-w)\rho\}$ carries out the role of a fan Γ.

Let's consider as an example the algebra A_2. It is easy to check that its star $\{(1-W)\alpha_3\} = 1-\Psi^0(A_2)$,



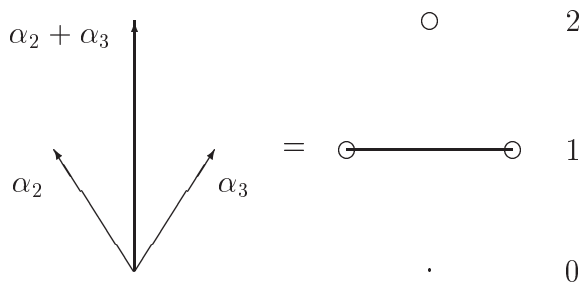
is nothing but a fan Γ, constructed on a set $\Xi(b(A_2))$ of representations of the Cartan subalgebra:

$$\Xi(b(A_2)) = \{(-1,2);(2,-1);(0,3);(+1,+1);(3,0)\}$$

Representations in Ξ are fixed by the eigenvalues of the basic generators $b_1, b_2 \in b(A_2)$.

For comparison we shall present two typical fans for injections of non-Abelian reductive subalgebras. Let $\widetilde{g} = A_1 \oplus u(1), g = A_2$. In this case the expression (31) can be easily and directly constructed. It defines a set of weights Γ:

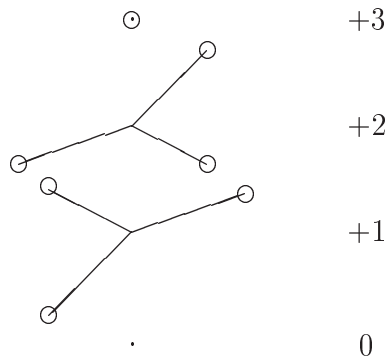
$\Gamma(A_1 \to A_2) =$



Thus, in the set of representations Ξ enter the spinor $d^1(A_1)$ and scalar $d^0(A_1)$ subrepresentations. The generator $u(1)$ accepts on the spaces of these subrepresentations the eigenvalues $+1$ and $+2$ correspondingly.

When the rank of algebra grows up the fan becomes more rich. For the injection $A_2 \oplus u(1) \to A_3$ we have:

$\Gamma(A_2 \oplus u(1) \to A_3) =$



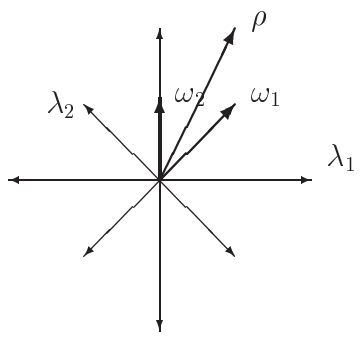
Here the set of subrepresentations of the algebra $sl(3)$ entering the fan contains both fundamental representations and also the trivial one.

3 *Example* $sl(2) \oplus u(1) \to so(5)$

To demonstrate the application of the factored formula (24) in details, we shall start with a very simple example.

Let's consider an injection $A_1 \oplus u(1) \to B_2$. Consider the simple roots of B_2 in the e-base

$$S(B_2) = \{\lambda_1 = e_1 - e_2, \lambda_2 = e_2\}$$



Here the fundamental weights are

$$\{\omega_1 = e_1, \omega_2 = \frac{1}{2}(e_1 + e_2)\},$$

and the Weil vector ρ is

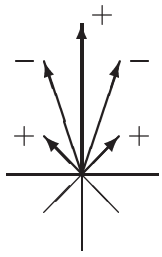
$$\rho = \frac{3}{2}e_1 + \frac{1}{2}e_2.$$

According to table 1 in the set $\Xi(A_1 \oplus u(1) \to B_2)$ we have four highest weights:

$$\Xi(A_1 \oplus u(1) \to B_2) = \{(0,0), (1,0), (2,1), (2,2)\}.$$

Thus the set $\Gamma(A_1 \oplus u(1) \to B_2)$ contains the weight diagrams $N^{\xi_1} = ([1], 1)$, $N^{\xi_2} = ([1], 3)$, $N^{\xi_s} = ([0], 4)$ of the representations of algebra $A_1 \oplus u(1)$, The generator $u(1)$ is normalized to have the integer eigenvalues:

$$\begin{aligned} \Gamma &= (\textstyle\bigcup_{\xi\in\Xi} N^{\xi}) \setminus \{0\} = \{\gamma^{(\mathrm{sign}\gamma)}\} = \\ &= \{(1,0)^{(+)}, (0,1)^{(+)}, (2,1)^{(-)}, (1,2)^{(-)}, (2,2)^{(+)}\}. \end{aligned}$$



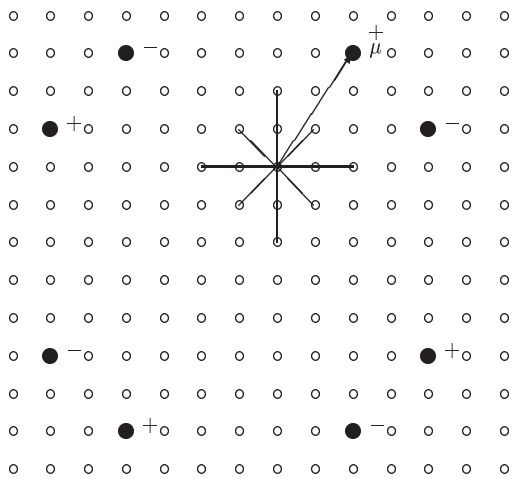
To simplify the following steps it is convenient to execute an additional decomposition of the anomalous weights diagram for subrepresentations $\widetilde{d}^{\mu}$. This decomposition is not unique, it is possible to pick an auxiliary vector $\varepsilon \in C$, and projections $\pi(\kappa)$ will turn out to be the scalar products

$$\langle(\kappa - \lambda), -\varepsilon\rangle = \pi(\kappa)$$

for each κ from the weight lattice P_g. The weight κ belongs to the floor $\pi(\kappa)$. The induced ordering of the components in Ψ^{μ} guarantees the consistent step by step application of the recursion formula (24). If $\Delta \setminus \widetilde{\Delta}$ does not contain

the positive roots orthogonal to the boundary of $\overline{C}$ the additional vector ε can be also located in the closure of C.

Let's consider an irreducible representation d^μ of B_2 with the highest weight $\mu = (5/2, 1/2)$.



On the picture above we have indicated the anomalous weights Ψ^μ of the representation $d^{(5/2e_1+1/2e_2)}$ that contains 8 vectors:

$$\begin{aligned}\Psi^{(5/2,1/2)} &= \{\psi^{\det(w)}\} = \\ &= \{(5/2,1/2)^{(+)}, (-1/2,7/2)^{(-)}, (5/2,-3/2)^{(-)}, \\ &(-5/2,7/2)^{(+)}, (-1/2,-9/2)^{(+)}, (-11/2,1/2)^{(-)}, \\ &(-5/2,-9/2)^{(-)}, (-11/2,-3/2)^{(+)}\}.\end{aligned}$$

This is the case when the decomposition can be simplified. It is possible to pick $\varepsilon = (1,1) \in \overline{C}$ so that $\langle \kappa, \varepsilon \rangle$ becomes proportional to eigenvalues of the generator $u(1)$ in the representation d^μ. While applying the factorized formula we want κ to be located inside the Weil chamber $\overline{\widetilde{C}}$. Thus in our example only the weight with a non-negative projection on $\lambda_1 = (1,-1)$ can have positive multiplicities n_κ. On the zero floor the result is trivial:

$$\widetilde{n}_{(5/2,1/2)} = n_{(5/2,1/2)} = 1,$$

$$\widetilde{n}_{(-1/2,7/2)} = -1.$$

On the following floor (called the "first") inside the chamber $\overline{\widetilde{C}}$ it is possible to find two points, $(5/2,-1/2)$ and $(3/2,1/2)$, where the formula (24) gives

nonzero values for $\widetilde{n}_\kappa$:

$$n_{(5/2,-1/2)} = 1, \quad n_{(3/2,1/2)} = 1.$$

Similarly at the following two levels we shall obtain

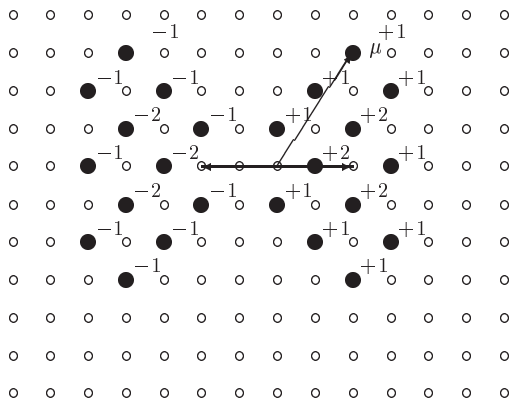
$$n_{(1/2,1/2)} = 1, \quad n_{(3/2,-1/2)} = 2 \quad \text{and} \quad n_{(1/2,-1/2)} = 2, \quad n_{(3/2,-3/2)} = 1.$$

Due to the symmetry of the weight diagram with respect to reflections these four levels give the information sufficient to write down the final result:

$$\begin{aligned}
&[2,1]_{\downarrow A_1 \oplus u(1)} = \\
&= \ ([2],3) \oplus ([1],2) \oplus ([3],2) \oplus 2([2],1) \oplus \\
&\quad \oplus([0],1) \oplus 2([1],0) \oplus ([3],0) \oplus 2([2],-1) \oplus \\
&\quad \oplus([0],-1) \oplus ([3],-2) \oplus ([1],-2) \oplus ([2],3).
\end{aligned}$$

Here numbers in square brackets are the Dynkin indices, and the last term in round brackets is the eigenvalue of the $u(1)$-generator.

Here is the full set of anomalous weights of the obtained subrepresentations as they are given by the formula (24). In the right hand side stand the highest weights, they completely define the set of subrepresentations and thus describe the branching:



Let's remark, that performing this reduction we do not need the anomalous weights other than inside the principal chamber $\overline{\widetilde{C}}$. Such additional property of the recurrence procedure appears only when the vectors of the

form $\{\gamma + \xi \mid \xi \in \overline{\widetilde{C}}, \gamma \in \Gamma\}$ do not reach the area of the anomalous weights of the subalgebra $\widetilde{g}$ in $P_g \setminus (P_g \cap \overline{\widetilde{C}})$.

In the above example the effectiveness of the reduction algorithm based on the factorized recursion formula (24) was demonstrated. The full set of calculations is rather simple and can be easily computerized. Non maximal regular injections and special injections can be treated similarly.

Chapter 2.

Relativistic symmetry and relativistic particles

4 *Induced representations*

Unitary representations play the distinguished role in quantum theory. For semisimple compact symmetry groups such representations are usually constructed applying the exponential map to the representations of the corresponding Lie algebras. When the group is noncompact or nonsemisimple the infinitesimal approach is ineffective. The most powerful representation method for nonsemisimple Lie groups is the so called *little group method.*

4.1 *Algebraic construction.*

Any Lie group G is universally supplied by (left D^L and right D^R) regular representation. The sets of irreducible subrepresentations in them are usually infinite. Our aim is to elaborate an effective method of reduction that will produce irreducible components of regular representation.

Consider the group G with the subgroup K . Let $D\,(K,V)$ be the representation of K on the space V. Consider on G the space of smooth functions $\mathrm{Fun}(G;V)$ with values in the space V. In $\mathrm{Fun}(G;V)$ let us extract the subspace $\mathrm{Fun}(G,K;D)$ of functions subject to *structure condition*:

$$f(gk) = D(k^{-1})f(g), \quad g \in G, k \in K. \tag{32}$$

$$\mathrm{Fun}(G,K;D) = \left\{ f \in \mathrm{Fun}(G;V) \mid f(gk) = D(k^{-1})f(g) \right\}.$$

Let X be the left factor space $X = G/K$. The structure condition (32) means that the values $f(g)$ in the points of the left class g_xK $(g \in x = g_xK)$ can be obtained when the values $f(g_x)$ for the representatives g_x are known: $f(g)\mid_{g=g_xk} = D(k^{-1})f(g_x)$.

Let us apply the operators of the left regular representation to the function $f(g) = f\,(g_xk) \in \mathrm{Fun}(G,K;D)$. It can be demonstrated that the property (32) is invariant with respect to the left regular representation D^L :

$$\begin{aligned} D^L\,(g')\,f(g) &= f(g'^{-1}g_xk) = D^L\,(g')\,D(k^{-1})f(g_x) \\ &= D(k^{-1})f(g'^{-1}g_x). \end{aligned} \tag{33}$$

Thus the restriction of D^L on the subspace $\mathrm{Fun}(G,K;D)$ form a representation of G.

Definition 4- 1 *: The subrepresentation of the left regular representation $D^L(G)$ acting on the subspace $\mathrm{Fun}(G,K;D)$ is called the representation $D_{K\uparrow G}$ induced by the representation $D(K)$.*

In the particular case $K=e$, $\dim V=1$ the space $\mathrm{Fun}(G,K;D)$ coincides with the space of scalar functions over G. Here the induced representation $D_{K\uparrow G}$ is equivalent to the regular representation: $D_{K\uparrow G}\approx D^L(G)$. Similarly when the representation $D(K)$ is trivial (for an arbitrary subgroup $K\subset G$) the induction leads to the quasiregular representation $D_{K\uparrow G}\approx D^{QL}(G)$. In the latter case the elements of the representation space have the property

$$f(gk)=f(g),\quad g\in G,k\in K.$$

Exercise 4- 1 *The structure condition (32) shows that we can realize the induced representation also on the space $\mathrm{Fun}(X;V)$ of smooth functions on the factor space $X=G/K$ with values in V. Let us correspond to each function $f\in\mathrm{Fun}(G,K;D)$ the function $\varphi\in\mathrm{Fun}(X;V)$ defined by the relation*

$$\varphi(x)=f(g_x). \tag{34}$$

Demonstrate that the spaces $\mathrm{Fun}(G,K;D)$ and $\mathrm{Fun}(X;V)$ are isomorphic.

Let ζ be the isomorphism constructed in the Exercise 1, $\zeta:\mathrm{Fun}(G,K;D)\longrightarrow\mathrm{Fun}(X;V)$. The representation $D^X_{K\uparrow G}:=\zeta D_{K\uparrow G}\zeta^{-1}$ is equivalent to $D_{K\uparrow G}$ and has $\mathrm{Fun}(X;V)$ as a representation space:

$$\begin{aligned}\zeta D_{K\uparrow G}(g)\zeta^{-1}\varphi(x) &= \zeta D_{K\uparrow G}(g)f(g_x)=\zeta f(g^{-1}g_x)=\\ &= \zeta f(g_{g^{-1}x}k(g^{-1},x))=\\ &= D\left(g_x^{-1}gg_{g^{-1}x}\right)\varphi\left(g^{-1}x\right).\end{aligned} \tag{35}$$

Here we have used the properties of the factor space decomposition of the group G:

$$\begin{aligned}g^{-1}g_x &= g_{g^{-1}x}k(g^{-1},x),\\ k(g^{-1},x) &= (g_{g^{-1}x})^{-1}g^{-1}g_x.\end{aligned}$$

Thus the induced representation $D^X_{K\uparrow G}$ transforms the elements φ of the space $\mathrm{Fun}(X;V)$ as follows:

$$\left(D^X_{K\uparrow G}(g)\varphi\right)(x)=D\left(g_x^{-1}gg_{g^{-1}x}\right)\varphi\left(g^{-1}x\right). \tag{36}$$

(In what follows we shall suppress the index X in $D^X_{K\uparrow G}$ when this will not lead to ambiguities.) It is not difficult to check that the representations constructed for different sets of representatives $\{g_x\}$ are equivalent.

Induced representations can be equally based on the right shifts, right factor space $K\backslash G$ and the regular representation $D^R(G)$.

4.2 *Unitarity.*

Below we shall see that the subgroup $K \subset G$ and the representation $D(K)$ can be chosen in such a way that the induced representation will be unitary and irreducible.

Firstly we shall consider the conditions guaranteeing the representation $D_{K\uparrow G}$ to be unitary. In the spaces $\mathrm{Fun}(G, K; D)$ and $\mathrm{Fun}(X; V)$ the corresponding scalar products will be constructed. This procedure is called the *Mackey method* and the corresponding representations – *induced by Mackey method.*

We start with the simple case when the group G is unimodular and the subgroup K is compact. $D(K, V)$ $\mathrm{Fun}(G, K; D)$ G $\mathrm{Fun}(G, K; D)$

$$(f_1, f_2)_F = \int_G (f_1(g), f_2(g))_V \, d\mu(g). \tag{37}$$

The representation $D_{K\uparrow G}$ on the space $\mathrm{Fun}(G, K; D)$ is evidently unitary.

In the integral (37) the argument is constant on the classes $x = g_x K$. This is why the factorization of the measure $d\mu(g)$ transforms the scalar product $(,)_F$ to the following form:

$$\begin{aligned}(f_1, f_2)_F &= \int d\mu(x) \int d\mu(k) \, (f_1(g_x k), f_2(g_x k))_V = \\ &= \int d\mu(x) \, (\varphi_1(x), \varphi_2(x))_V . \end{aligned} \tag{38}$$

When the group K is compact the scalar product (37) loose its meaning. Nevertheless the expression (38) remains valid as a definition of the scalar product (φ_1, φ_2) in the space $\mathrm{Fun}(X; V)$.(In this case the unitarity of $D^X_{K\uparrow G}$ is guaranteed by the unitarity of the representation $D(K, V)$.)

In the case of nonunimodular groups G and K the unitary representation $D(K)$ is used to construct the nonunitary one, $C(K)$,

$$C(k) = \sqrt{\frac{\Delta_K(k)}{\Delta_G(k)}} D(k), \tag{39}$$

where $\Delta_G$and Δ_K are the modules of the corresponding groups. This representation is then used to induce (applying the standard procedure) the representation $C_{K\uparrow G}$ on the space $\mathrm{Fun}\,(G,K;C)$.

Exercise 4- 2 *Check the unitarity of* $C_{K\uparrow G}$ *on* $\mathrm{Fun}\,(G,K;C)$ *with the scalar product (38) and the quasi-invariant measure* $d\mu\,(g)$.

5 *Induced representations, relativistic symmetry*

5.1 *Little group method*

In this subsection we shall study how to construct induced unitary irreducible representations for Lie groups containing nontrivial normal subgroup. It was elaborated by E.P. Wigner [23] and the main goal was to obtain the full spectrum of irreducible representations for the group Poincare. In the 50's of the 20-th century G. Mackey [24] performed the detailed study of induced representations and in particular the little group method. The latter was widely used in the group theory and also in relativistic quantum physics.

Let G be a connected Lie group. The Levi-Maltzev theorem tells us that it can be presented as a semidirect product $G \approx S \rhd N$ with the semisimple subgroup S and the maximal solvable subgroup N. Let D_N be an irreducible representation of the normal subgroup. To any element $g \in G$ let us relate the representation constructed from D_N according to the rule

$$D_N^g\,(n) = D_N\left(g^{-1}ng\right)$$

The set of representations $\{D_N^g \mid g \in G\}$ is called *the orbit* of the representation D_N.

Among the elements D_N^g of the orbit let us find those that are equivalent to D_N, for example such as D_N^m with $m \in N$. Let us select all the elements $h \in G$ for which $D_N^h \approx D_N$. It is easy to check that these elements $\{h\}$ form the subgroup $H \subset G$. This subgroup is called *the little group of the representation* D_N. Obviously the radical N is a normal subgroup in H. The factor group H/N is called *the little cogroup of the representation* D_N.

When the initial representation is taken to be any other member D_N^g of the orbit $\{D_N^g\}$ the corresponding little group H' is conjugate to the initial one: $H : H' = gHg^{-1}$. All the little groups related to the same orbit are equivalent.

Theorem 5- 1 *Let $G \approx S \rhd N$ be the levi-Maltzev decomposition of the Lie group G. Consider an irreducible (one-dimensional) representation D_N of the normal subgroup N. Find its little group H and little cogroup $R = H/N$. Consider an irreducible representation D_R of R and the representation B_H of the form*

$$B_H(nh_r) = D_N(n) D_R(r). \tag{40}$$

The induced representation $B_{H\uparrow G}$ is irreducible and any irreducible representation of G can be obtained by the same algorithm. For each orbit $\{D_N^g\}$ the class of equivalent irreducible representations of G is thus associated. Any unitary irreducible representation of G can be obtained as $B_{H\uparrow G}$ using unitary representations D_N and D_R.

In our lectures we shall consider the most interesting algebraic part of the proof, for more details see [6], Ch. 17, Sect. 1.

Proof. Suppose we have obtained a representation $D_{N\uparrow H} \equiv C_H$. According to the subgroups induction theorem its restriction $C_{H\downarrow N}$ can be decomposed into a direct integral

$$C_{H\downarrow N} = D_{N\uparrow H\downarrow N} = \int_{Y'} d\mu(y') D^{y'}_{N_{y'}\downarrow L_{y'}\uparrow N} \quad (Y' = N\backslash H/N).$$

As far as the subgroup N is normal in H we have the equivalence relation $N_{y'} \approx N \approx L_{y'}$ and the induced representation has the decomposition

$$C_{H\uparrow N} = \int_{Y'} d\mu(y') D^{y'}_N. \tag{41}$$

All the representations $D^{y'}_N$ are equivalent to the irreducible and unitary D_N. The property (41) is obviously true also for any irreducible component $C^{(n)}_H \equiv B_H$ of the representation C_H.

Let us show that the induced representation $B_{H\uparrow G}$ is irreducible. Consider the space $Y = H\backslash G/H$, find the groups $H_y = g_y H g_y^{-1}$, $L_y = H_y \cap H$ and construct the representations $B^y_{H_y}$ and $B^y_{H_y}\left(g_y h g_y^{-1}\right) = B(h)$. It is evident that any group H_y contains the normal subgroup N. The same is true also for any group L_y. For nontrivial classes $y \in Y$ the absence of nontrivial intertwining operators for the restrictions $B^y_{H_y\downarrow L_y\downarrow N}$ $B_{H\downarrow L_y\downarrow N}$ is sufficient to provide the trivialization of the space, $\zeta\left[B^y_{H_y\downarrow L_y}, B_{H\downarrow L_y}\right] = 0$. Taking

into account that for $B_{H\downarrow N}$ the decomposition (41) is valid we see that any representation $B_{H_y\downarrow N}$ can also be decomposed

$$B^{y}_{H_y\downarrow N} = \int D^{(y)y'}_{N} d\mu\left(y'\right). \tag{42}$$

The representations $D^{(y)y'}_{N}$ are formed from $D^{y'}_{N} \approx D_N$ using the elements $g_y \notin H$. Consequently, for $y \neq \{H\}$ no equivalent representations can be found in the set $\left\{\left(D^{(y)y'_1}_{N} D^{y'_2}_{N}\right)\right\}$. The irreducibility of $B_{H\uparrow G}$ is proved.

Thus an irreducible unitary representation of the group G can be induced by an irreducible component B_H of the representation $D_{N\uparrow H}$. Let us now demonstrate that for the representation B_H the relation (40)is true. The little group H contains the normal subgroup N. It follows that H, for the same reasons as G, can be presented as a semidirect product $H \approx R \rhd N$. Elements r of the group R can serve as representatives of classes H/N. Notice that all the corresponding factors $n\left(r_1, r_2\right)$ are trivial.

The representation C_H acts on the space $F\left(H, N; D\right)$. Any element $h \in H$ can be written as a product $h = nr$. This means that due to the structure condition an arbitrary function $f \in F\left(H, N; D\right)$ is completely defined by its values on the factor group R. Thus it is sufficient to consider the action of the operators of $C_H\left(h\right)$ on the space $F\left(R, N; D\right)$:

$$\begin{aligned} C_H\left(h\right) f\left(r_1\right) &= D_{N\uparrow H}\left(h = nr\right) f\left(r_1\right) = f\left(r^{-1} n^{-1} r_1\right) = \\ &= f\left(r^{-1} r_1 \chi_{r_1}\left(n^{-1}\right)\right) = D^{-1}_{N}\left(\chi_{r_1}\left(n^{-1}\right)\right) f\left(r^{-1} r_1\right) = \\ &= D^{r_1}_{N}\left(n\right) f\left(r^{-1} r_1\right). \end{aligned} \tag{43}$$

The representation D_N is irreducible. Consequently it is one-dimensional. According to the definition of the small group the representation $D^{r_1}_{N}\left(n\right)$ is equivalent to D_N, in fact it coincides with D_N because it is also one-dimensional $\dim D^{r_1}_{N}\left(n\right) = 1$. Thus we have

$$C_H\left(h = nr\right) f\left(r_1\right) = D_N\left(n\right) f\left(r^{-1} r_1\right). \tag{44}$$

It follows that the operators of $C_H\left(n\right)$ act by multiplying the element of the representation space by a number. At the same time the operator $C_H\left(r\right)$ performing the left shift of scalar functions f is an element of the left regular representation of the group R. This means that in $F\left(H, N; D\right)$ the subspace irreducible with respect to $C_{H\downarrow R}$ is also C_H-irreducible. In other words to

extract an irreducible component B_H we are to perform the similar procedure with the left regular representation of the group R.

Suppose that problem is solved and the irreducible unitary representation $D(R,V)$ is constructed. Let B_H be an irreducible component of C_H such that $B_{H\downarrow R} \approx D(R,V)$. For B_H the formula is also valid

$$D_N(n)\, D_R(r)\, v = B_H(nr)\, v \qquad (v \in V). \tag{45}$$

It follows that the operators $B_H(h)$ have the structure (40).

Any irreducible representation D_G can be extracted as a component from the one that was induced by its restriction $D_{G\downarrow N}$.

Exercise 5- 1 *Demonstrate that any unitary irreducible representation of the group G can be presented in the form $B_{H\uparrow G}$ where B_H is defined as in (45).*

Now let us consider again the representation $D(G,W)$ and its restriction $D_{G\downarrow N}$. Obviously for any irreducible component $D^{(n)}_{G\downarrow N}$ in the representation B_H (formed according to the relation (45)) the induced representation $B_{H\uparrow G}$ is equivalent to D_G.

Let $D^{(n)}_{G\downarrow N}$ be defined on the space $W^n \subset W$. As far as D_G is irreducible any vector $w^m \in W^m$ (corresponding to the irreducible component $D^{(m)}(G\downarrow N, W^m)$) can be presented as $D_G(g)\, w^n$. It follows that any representation $D^{(m)}_{G\downarrow N}$ is equivalent to $D^{(n)g}_{G\downarrow N}$. To obtain the irreducible representations of G (from each class of equivalent ones) it is sufficient to take one representative D_N in every orbit of the irreducible representation of N. ■

Remark 5- 1 *We have not used the semisimplicity of the group S. The theorem above remains true if G is isomorphic to a semidirect product of a solvable group N and a reductive group K.*

Example 5- 1 *Here we shall consider the simplest case by constructing the unitary irreducible representations of the two-dimensional group of motion $\mathbf{E}(2)$. In this case despite the fact that the group $\mathbf{E}(2)$ is solvable the Levi decomposition can be written for it: $\mathbf{E}(2) \approx U(1) \rhd \mathbf{P}(2)$, the factor $U(1)$ is Abelian.*

Let $\mathbf{b}$ and $\mathbf{t}$ be the two-dimensional real vectors from the spaces $P(2)$ and $P(2)^{(*)}$ correspondingly. The vector $\mathbf{t}$ parameterizes the irreducible unitary representations of the translation group $\mathbf{P}(2)$:

$$D^{\mathbf{t}}_{\mathbf{P}(2)}(\tau_{\mathbf{b}}) = \exp i(\mathbf{t},\mathbf{b}). \tag{46}$$

This group is normal in $\mathbf{E}(2)$ *and its topological space* $P(2)$ *is a* $\mathbf{E}(2)$*-module with respect to the adjoint transformations*

$$\chi_{(g=\tau_{\mathbf{c}}u)}:\left\{\begin{array}{c}\tau_{\mathbf{b}}\rightarrow g^{-1}\tau_{\mathbf{b}}g=u^{-1}\tau_{\mathbf{b}}u=\tau_{(u^{-1}\mathbf{b})},\\ \mathbf{b}\rightarrow u^{-1}\mathbf{b},\end{array}\right| \tag{47}$$

Here u*'s act on the vectors* $\mathbf{b}\in P(2)$ *as rotations in the defining representation. On the dual space* $P(2)^{(*)}\equiv T(2)$ *the dual* $\mathbf{E}(2)$*-module is formed:*

$$\chi_{(g=\tau_{\mathbf{c}}u)}:\mathbf{t}\rightarrow u^{-1}\mathbf{t}. \tag{48}$$

In the space $T(2)$ *the orbits of the group* $\mathbf{E}(2)$ *are circles with the radius* r*. In the set of orbits there are two classes corresponding to* $r>0$ *and* $r=0$*. To each orbit* $Orb\mathbf{E}(2)$ *in* $T(2)$ *an orbit* $\left\{\left(D^{\mathbf{t}}_{\mathbf{P}(2)}\right)^{g}\right\}(\mathbf{t}\in OrbO(2))$ *of the representation* $D^{\mathbf{t}}_{\mathbf{P}(2)}$ *can be attributed:*

$$\left(D^{\mathbf{t}}\right)^{g}(\tau_{\mathbf{b}})=D^{\mathbf{t}}\left(g^{-1}\tau_{\mathbf{b}}g\right)=D^{\mathbf{t}}\left(u^{-1}\tau_{\mathbf{b}}u\right)=D^{\mathbf{t}}\left(\tau_{(u^{-1}\mathbf{b})}\right)=D^{\left(u^{T}\mathbf{t}\right)}(\tau_{\mathbf{b}}). \tag{49}$$

Consider the case $r>0$*.Here the stability subgroup* H *(that is the little group of the representation) coincides with* $\mathbf{P}(2)$ *(see the relation (48)) and (49)). The little cogroup is trivial. The representation* $B^{\mathbf{t}}_{H}$ *has the form*

$$B^{\mathbf{t}}_{H}(\tau_{\mathbf{b}})=D^{\mathbf{t}}_{\mathbf{P}(2)}(\tau_{\mathbf{b}}). \tag{50}$$

Induced representation will be constructed on the space of scalar functions whose arguments are in the homogeneous space $X\approx\mathbf{E}(2)/H\approx U(1)\approx S^{1}$*:*

$$\begin{aligned}&B^{\mathbf{t}}_{H\uparrow\mathbf{E}(2)}(\tau_{\mathbf{b}}u_{\alpha})\varphi\\ =\;&D^{\mathbf{t}}_{\mathbf{P}(2)}\left(u_{\beta}^{-1}\tau_{\mathbf{b}}u_{\beta}\right)\varphi(\beta-\alpha)=D^{\mathbf{t}}_{\mathbf{P}(2)}\left(\tau_{\left(u_{\beta}^{-1}\mathbf{b}\right)}\exp\right)\varphi(\beta-\alpha)=\\ =\;&\exp i\left(\left(u_{\beta}\mathbf{t}\right),\mathbf{b}\right)\varphi(\beta-\alpha),\end{aligned} \tag{51}$$

Here $\alpha,\beta\in[0,2\pi)$ *and* u_{β} *play the role of representatives for the classes* $x\approx S^{1}$*.*

When the orbit has $r=0$ *the group* $\mathbf{E}(2)$ *itself is the little group* H *(see the formula (49)). The little cogroup* R *is isomorphic to* $U(1)$*. The irreducible representations of* R *can be presented in the form*

$$D^{j}_{R}(u_{\alpha})=\exp ij\alpha\,(\alpha\in[0,2\pi),\quad j=0,\pm1,\pm2,...). \tag{52}$$

As far as the group $\mathbf{E}(2)$ *coincides with the little group the induced representation sought for is equivalent to* B_H*:*

$$B^j_{H\uparrow\mathbf{E}(2)}(g) \approx B^j_H(g = \tau_{\mathbf{b}} u_\alpha) = D^j_R(u_\alpha) = \exp ij\alpha. \tag{53}$$

We have obtained the full spectrum of irreducible unitary representations of the group $\mathbf{E}(2)$*.*

5.2 *Poincare group irreducible representations*

Our main task is to construct irreducible unitary representations of the Poincare group. In it the composition is defined by the multiplication in the Lorentz group $\mathbf{\Lambda}$, the translations $\mathbf{P}$ and the automorphisms χ_Λ:

$$\begin{aligned} \Pi'\Pi &= (\Lambda', \tau_{a'})(\Lambda, \tau_a) = (\Lambda'\Lambda, \chi_\Lambda(\tau_{a'})\tau_a) = \\ &= (\Lambda'\Lambda, \tau_{\Lambda^{-1}a'}\tau_a) = \left(\Lambda'\Lambda, \tau_{(\Lambda^{-1}a'+a)}\right). \end{aligned} \tag{54}$$

In the little group method it is supposed that the group is connected. Thus we start with the connected component of the Poincare group – the proper ortochronous Poincare group $\mathbf{\Pi}^\uparrow_+ \approx \mathbf{\Lambda}^\uparrow_+ \rhd \mathbf{P}$.

In the quantum theory phases of the wave functions are not observable and unitary representations of $\mathbf{\Pi}$ must be projective. For the Poincare group (as well as for most of groups of motions) unitary projective representations are equivalent to the linear representations of the universal covering group for $\mathbf{\Pi}$. (The Galilee group presents an interesting counterexample: to construct its projective representations it is insufficient to pass to the universal covering, the group must be essentially enlarged.) This universal covering is called the *quantum mechanical Poincare group* and we shall denote it by $\mathbf{\Pi}$,

$$\mathbf{\Pi} \equiv \widehat{\mathbf{\Pi}}^\uparrow_+ \approx SL(2,C)_R \rhd \mathbf{P} \equiv \mathbf{L} \rhd \mathbf{P}. \tag{55}$$

Consider $\mathbf{\Pi}$ as $\mathbf{\Pi}$-module with respect to the adjoint transformations. The normal subgroup $\mathbf{P}$ also becomes a $\mathbf{\Pi}$-module. Evidently $\mathbf{P}$ acts trivially on itself. The action of $\mathbf{L}$ on $\mathbf{P}$ is defined by the automorphisms χ_L. We shall present explicitly the corresponding transformations on the space P of $\mathbf{P}$ and also of the dual space $P^{(*)} \ni p$ that has the natural structure of the dual $\mathbf{\Pi}$-module:

$$\begin{aligned} L &: \tau_a \to \chi_L(\tau_a) = L^{-1}\tau_a L = \tau_{\left(\Lambda_L^{-1}a\right)}, \\ L &: a \to \Lambda_L^{-1}a, \qquad L : p \to \Lambda_L p, \\ \tau_b &: a \to a, \qquad \tau_b : p \to p. \end{aligned} \tag{56}$$

Notice that the $\mathbf{L}$-module $P^{(*)}$ is equivalent to the $\mathbf{L}$-module M_4.

For the orbits of $\mathbf{\Pi}$ on $P^{(*)}$the classification scheme is the same as when $\mathbf{L}$ acts on the Minkowski space M_4:

$$\begin{array}{rrcl} 1. & \mathcal{O}_m^{\dot{+}} & \equiv & \left\{p \mid p^2 = m^2; \quad m^2 > 0; \quad p_0 > 0\right\}; \\ 2. & \mathcal{O}_m^{-} & \equiv & \left\{p \mid p^2 = m^2; \quad m^2 > 0; \quad p_0 < 0\right\}; \\ 3. & \mathcal{O}_{im} & \equiv & \left\{p \mid p^2 = -m^2; \quad m^2 > 0\right\}; \\ 4. & \mathcal{O}_0^{\dot{+}} & \equiv & \left\{p \mid p^2 = 0; \quad p_0 > 0\right\}; \\ 5. & \mathcal{O}_0^{-} & \equiv & \left\{p \mid p^2 = 0; \quad p_0 < 0\right\}; \\ 6. & \mathcal{O}_0^{0} & \equiv & \{p = (0,0,0,0)\}. \end{array} \tag{57}$$

In the corresponding $\mathbf{L}$-homogeneous spaces the stability subgroups are:

1. $SU(2)$ for $\mathcal{O}_m^{\dot{+}}$and $\mathcal{O}_m^{-}$,

2. $SL(2,R) \approx SU(1,1)$ for $\mathcal{O}_{im}$,

3. $\widetilde{\mathbf{E}}(2)$ (the double covering of $\mathbf{E}(2)$) for $\mathcal{O}_0^{\dot{+}}$ and $\mathcal{O}_0^{-}$,

4. the Lorentz group $\mathbf{L}$ for $\mathcal{O}_0^0$.

On the vectors $p \in P^{(*)}$ the subgroup $\mathbf{P}$acts trivially, thus the stability subgroups for the $\mathbf{\Pi}$-homogeneous spaces are

1. $\mathcal{O}_m^{\dot{+}}$ and $\mathcal{O}_m^{-} - SU(2) \rhd \mathbf{P}$, with the standard vector $p = (m,0,0,0)$;
2. $\mathcal{O}_0^{\dot{+}}$ and $\mathcal{O}_0^{-} - \widetilde{\mathbf{E}}(2) \rhd \mathbf{P}$, with the standard vector $p = (1,0,0,1)$;
3. $\mathcal{O}_{im} - -SU(1,1) \rhd \mathbf{P}$, with the standard vector $p = (0,m,0,0)$;
4. $\mathcal{O}_0^0 - -\mathbf{\Pi}$.

Now we start the construction of the irreducible representations for $\mathbf{\Pi}$. Consider an irreducible unitary representation of $\mathbf{P}$,

$$D_P(\tau_a) = \exp i(p \cdot a). \tag{58}$$

Here the vector p is an index of the representation. It obviously belongs to the dual $\mathbf{\Pi}$-module $P^{(*)}$. Notice that in $D_P$any translation generator is realized by the multiplication operator, it multiplies the representation space

vector by the component of the vector p. Let us construct the orbit of the representation D_P:

$$\begin{aligned} D^{\Pi}(\tau_a) &= D\left(\Pi^{-1}\tau_a\Pi\right) = D\left(L^{-1}\tau^{-1}\tau_a\tau L\right) = D\left(L^{-1}\tau_a L\right) = D\left(\chi_L(\tau_a)\right) = \\ &= D\left(\tau_{\left(\Lambda_L^{-1}a\right)}\right) = \exp i\left(p\cdot\left(\Lambda_L^{-1}a\right)\right) = \exp i\left(\left(\Lambda_L p\right)\cdot a\right). \end{aligned} \tag{59}$$

This expression demonstrates that the variety of the irreducible representations $\left\{D_P^{\Pi}\right\}$ are indexed by the parameters p belonging to an orbit of the Poincare group in the space $P^{(*)}$.

Let $p \in \mathcal{O}_m^{\dot{-}}$. Fix the initial representation D_p with an index p (here the standard vector is $p = (m, 0, 0, 0)$. Then it follows from (59) that the small group H of D_p is isomorphic to the stability subgroup of the standard vector in the orbit $\mathcal{O}_m^{\dot{-}}$. The small subgroup R coincides with $SU(2)$:

$$H \approx SU(2) \rhd \mathbf{P}, \qquad R \approx \widehat{\mathbf{R}} \approx SU(2). \tag{60}$$

We know the full system of unitary irreducible representations $D^J_{\widehat{\mathbf{R}}}$ for the group $SU(2)$. They are enumerated by the spin J. Using the operators $D^J_{\widehat{\mathbf{R}}}(r)$ and $D^{\overset{o}{p}}_{\mathbf{P}}(\tau)$ let us construct the irreducible unitary representation B_H according to the formula (40):

$$B_H^{J\,\overset{o}{p}}(\tau h_r) = D^{\overset{o}{p}}_{\mathbf{P}}(\tau)\, D^J_{\widehat{\mathbf{R}}}(r). \tag{61}$$

The irreducible unitary representation $B^{J\,\overset{o}{p}}_{H\uparrow\Pi}$ of the group Poincare has $\mathrm{Fun}(X, V)$ as a representation space. This is the space of square integrable functions φ on $X = \mathbf{\Pi}/H \approx \mathbf{L}/\widehat{\mathbf{R}}$ and with values in the space V – the representation space of $B_H^{J\,\overset{o}{p}}$. In face of the formula (61) one can identify the space $V^{J\,\overset{o}{p}}$ with the representation space V^J of $D^J_{\widehat{\mathbf{R}}}$ (notice that $\dim D^{\overset{o}{p}}_{\mathbf{P}} = 1$). The factor space X is equivalent to the orbit $\mathcal{O}_m^{\dot{-}}$ (or $\mathcal{O}_m^{-}$ according to the classification (57)).

As it was noticed above each representation induced by $\left(D^{\overset{o}{p}}_{\mathbf{P}}(\tau)\right)^{\Pi} D^J_{\widehat{\mathbf{R}}}(r)$ is equivalent to $B^{J\,\overset{o}{p}}_{H\uparrow\Pi}$. Consequently the set $\left\{B^{J\,\overset{o}{p}}_{H\uparrow\Pi}\right\}$, $\quad p \in \mathcal{O}_m^{\dot{-}}$ consists of Poincare group representations equivalent to $B^{J\,\overset{o}{p}}_{H\uparrow\Pi}$. Thus the induced representation is characterized by the indices $(J, m, +)$ (or shortly by (J, m)).

Consider the induced representation $B^{J\,m}_{H\uparrow\Pi}$ realized in the space of functions on $\mathcal{O}^{\div}_{m}$ with values in V^J:

$$B^{J\,m}_{H\uparrow\Pi}\left(\Pi\right)\varphi\left(p\right)=B^{J\,\overset{o}{p}}\left(\Pi_p^{-1}\Pi\Pi_{\left(\Pi^{-1}p\right)}\right)\varphi\left(\Pi^{-1}p\right). \tag{62}$$

Let us fix the representatives of classes Πp that form the argument of the operators $B^{J\,\overset{o}{p}}\left(\Pi\right)$. For our purposes it is convenient to choose these representatives equal to the boosts L_p – the hermitian matrices from the group $SL\left(2,C\right)_R$ such that the following relation holds: $L_p\left(\overset{o}{p}\sigma\right)L_p=\left(p\sigma\right)$. We can redress the argument of $B^{J\,\overset{o}{p}}$ using the properties of factors and the formula (56):

$$\begin{aligned} L_p^{-1}\Pi L_{\left(\Pi^{-1}p\right)} &= L_p^{-1}\tau_a L L_{\left(L^{-1}\tau_a^{-1}p\right)}=L_p^{-1}\tau_a L_p L_p^{-1} L L_{\left(L^{-1}p\right)}= \\ &= \tau_{\left(\Lambda_p^{-1}a\right)}r\left(L,L^{-1}p\right). \end{aligned} \tag{63}$$

We can substitute this expression into the relation (62) and simplify the argument of φ according to the $\mathbf{\Pi}$-module structure of P:

$$\begin{aligned} B^{J\,m}_{H\uparrow\Pi}\left(\Pi\right)\varphi\left(p\right) &= B^{J\,\overset{o}{p}}\left(\tau_{\left(\Lambda_p^{-1}a\right)}r\left(L,L^{-1}p\right)\right)\varphi\left(L^{-1}\tau_a^{-1}p\right)= \\ &= D^{\overset{o}{p}}_{P}\left(\tau_{\left(\Lambda_p^{-1}a\right)}\right)D^{J}_{\widehat{\mathbf{R}}}\left(r\left(L,L^{-1}p\right)\right)\varphi\left(L^{-1}p\right)= \\ &= \exp i\left(\overset{o}{p}\cdot\left(\Lambda_p^{-1}a\right)\right)D^{J}_{\widehat{\mathbf{R}}}\left(r\left(L,L^{-1}p\right)\right)\varphi\left(L^{-1}p\right)= \\ &= \exp i\left(p\cdot a\right)D^{J}_{\widehat{\mathbf{R}}}\left(r\left(L,L^{-1}p\right)\right)\varphi\left(L^{-1}p\right). \end{aligned} \tag{64}$$

The functions φ_p describe the states of the quantum relativistic elementary particle. Each space $\mathrm{Fun}\left(X,V\right)\approx\mathrm{Fun}\left(\mathcal{O},V\right)$ of the irreducible representation $B^{J\,m}_{H\uparrow\Pi}$ bears the indices $\left(J,m,\pm\right)$ and thus the elementary particles are classified with respect to their space-time properties.

It is worth reminding that $p\in P$ in (58) realizes the translation generator in the representation D_p. The argument of φ_p belongs to $\mathcal{O}^{\div}_{m}$ that is to the factor space $X\approx\mathbf{\Pi}/H$. Let us check that here p also has the meaning of particle impulse. Transform the function φ_p by the operator $B^{J\,m}_{H\uparrow\Pi\downarrow P}$:

$$B^{J\,m}_{H\uparrow\Pi\downarrow P}\left(\tau_a\right)\varphi\left(p\right)=\exp i\left(p\cdot a\right)\varphi\left(p\right). \tag{65}$$

Notice that the eigenvalue of the momentum operator in the representation $B^{J\,m}_{H\uparrow\Pi}$ coincides with the argument of the state function. The physical meaning of vectors in the $\mathbf{\Pi}$-module $P^{(*)}$ is thus found to be the impulses of the

relativistic quantum objects. The invariant m used to classify the orbits in $P^{(*)}$ is obviously the mass of a particle.

To find the meaning of the index J it is sufficient to consider the transformation performed by $B^{J\,m}_{H\uparrow\Pi\downarrow\widehat{\mathbf{R}}}$ in the subspace of particle state vectors in the rest frame:

$$B^{J\,m}_{H\uparrow\Pi\downarrow\widehat{\mathbf{R}}}\left(r\right)\varphi\left(\overset{o}{p}\right)=D^{J}_{\widehat{\mathbf{R}}}\left(r\right)\varphi\left(\overset{o}{p}\right). \tag{66}$$

The obtained transformation rule signifies that the function $\varphi\left(\overset{o}{p}\right)$ describes a particle with the spin J. Summing up we conclude that the classification of particles with respect to the space-time symmetry properties is performed by indicating their spin, mass and the sign of energy.

Remark 5- 2 *The set of vectors $\varphi\left(p\right)$ can be considered as the wave function of a quantum relativistic particle. In this case the ensemble of observables is limited by the space-time symmetry. It is supposed that $\varphi\left(p\right)$ is defined on the spectrum of values (of observables) and $\varphi\left(p,m,J\right)$ is nonzero only when m and J are the indices of the representation $B^{J\,m}\left(\Pi\right)$. The index J corresponds to an eigenvalue of the fourth order Cazimir operator $B\left(C_4\right)$ of the Poincare algebra:*

$$B\left(C_2C_4\right)\varphi\left(p\right)=-4J\left(J+1\right)\varphi\left(p\right). \tag{67}$$

Exercise 5- 2 *Prove the relation (67).*

The representation $B^{J\,m}_{H\uparrow\Pi}$ is unitary with respect to the scalar product:

$$\left(\varphi_1,\varphi_2\right)_F=\int_{\mathcal{O}^{\div}_{m}}d\mu\left(p\right)\left(\varphi_1\left(p\right),\varphi_2\left(p\right)\right)_{V^J}. \tag{68}$$

Here $d\mu\left(p\right)$ is the invariant measure on the hyperboloid $\mathcal{O}^{\div}_{m}$:

$$\left(\varphi_1,\varphi_2\right)_F=\int_{P(3)}\frac{d^3p}{2\sqrt{p^2+m^2}}\left(\varphi_1\left(p\right),\varphi_2\left(p\right)\right)_{V^J}=\int_{P(3)}\frac{d^3p}{2p_0}\varphi^{\dagger}_1\left(p\right)\varphi_2\left(p\right), \tag{69}$$

the momentum p in the integrand has the components $\left(\sqrt{p^2+m^2},p^1,p^2,p^3\right)$.

In the expression (64) the representation operator $B^{J\,m}_{H\uparrow\Pi}\left(\Pi\right)$ is such that a state function transforms as a spinor only in the rest frame. To expose the

spinor properties of $\varphi(p)$ for any values of $p$let us change the basis in the space Fun $\left(X, V^J\right)$. First of all let us rewrite the representation action:

$$B^{J\,m}_{H\uparrow\Pi}(\Pi)\,\varphi(p) = \exp i\,(pa)\,D^J_{\widehat{\mathbf{R}}}\left(L_p^{-1} L L_{(L^{-1}p)}\right)\varphi\left(L^{-1}p\right). \tag{70}$$

We shall extend the representation $D^J_{\widehat{\mathbf{R}}}$ of the small cogroup to the representation $D^{(J,0)}_{\widehat{\mathbf{R}}}$ of the group $\mathbf{L}$. (The restriction $D^{(J,0)}_{L\downarrow\widehat{\mathbf{R}}}$ coincide with $D^J_{\widehat{\mathbf{R}}}$ and the representation spaces are the same: $V^{(J,0)} \approx V^J$.) In (70) the operator $D^J_{\widehat{\mathbf{R}}}(r)$ can be considered as such a restriction to $\widehat{\mathbf{R}}$ of the corresponding representation operator of the group $\mathbf{L}$:

$$\begin{aligned} B^{J\,m}_{H\uparrow\Pi}(\Pi) \quad & \varphi(p) = \exp i\,(pa)\,D^{(J,0)}_L\left(L_p^{-1} L L_{(L^{-1}p)}\right)\varphi\left(L^{-1}p\right) = \\ = \quad & \exp i\,(pa)\,D^{(J,0)}_L\left(L_p^{-1}\right) D^{(J,0)}_L(L)\,D^{(J,0)}_L\left(L_{(L^{-1}p)}\right)\varphi\left(L^{-1}p\right) \end{aligned} \tag{71}$$

Introduce the new basic functions

$$\psi(p) \equiv D^{(J,0)}_L(L_p)\,\varphi(p). \tag{72}$$

In such a basis we shall obtain the operators of the induced representation will take the form

$$B^{J\,m}_{H\uparrow\Pi}(\Pi)\,\psi(p) = \exp i\,(pa)\,D^{(J,0)}_L(L)\,\psi\left(L^{-1}p\right). \tag{73}$$

In this presentation the relativistic spinor properties of the function $\psi(p)$ are evident. The basis (72) is called the *spinor basis*.

The scalar product (69) can be rewritten in the spinor basis:

$$\begin{aligned} (\psi_1, \psi_2)_F \quad &= \int \frac{d^3p}{2p_0}\psi_1^\dagger(p)\,D^{\dagger -1}(L_p)\,D^{-1}(L_p)\,\psi_2(p) = \\ &= \int \frac{d^3p}{2p_0}\psi_1^\dagger(p)\,D^{-1}\left(L_p \frac{\left(\sigma \overset{o}{p}\right)}{m} L_p^\dagger\right)\psi_2(p) = \\ &= \int \frac{d^3p}{2p_0}\psi_1^\dagger(p)\,D\left(\frac{\sigma \overset{o}{p}}{m}\right)\psi_2(p). \end{aligned} \tag{74}$$

Here we have used the standard properties of the Pauli matrices and boosts in $\mathbf{L}$.

The construction presented above for the orbits $\mathcal{O}_m^{\dot{+}}$ can be applied in the case of $\mathcal{O}_m^-$ almost without changes (these orbits have the same stability group). The formulas (64) and (73) describe the behaviour of the relativistic elementary particles states with the rest mass $m \neq 0$ and spin J in the case of positive energy as well as in the case of negative one. We shall not consider here the representations corresponding to the orbits $\mathcal{O}_{im}$ and $\mathcal{O}_0^0$ (that is to particles with an imaginary mass and particles with constant zero momentum), though they also play some role in the theory of elementary particles (see [25] for details).

Let us consider the orbits $\mathcal{O}_0^{\dot{+}}$ and $\mathcal{O}_0^-$ with the stability subgroup $\widetilde{\mathbf{E}}(2) \triangleright \mathbf{P}$ and the standard vectors $\overset{o}{p} = (1,0,0,1)$ and $\overset{o}{p} = (-1,0,0,-1)$. Consider the representation $D_{\mathbf{P}}^{\overset{o}{p}}$ (see the relation (58)) and choose the standard vector $\overset{o}{p} = (1,0,0,1)$. Using the relations (59) and (52) we can find the little group and cogroup:

$$H \approx \widetilde{\mathbf{E}}(2) \triangleright \mathbf{P}, \qquad \mathbf{R} \approx \widetilde{\mathbf{E}}(2). \tag{75}$$

Earlier we have obtained the full set of irreducible unitary representation for the group $\mathbf{E}(2)$. Poincare group $\mathbf{\Pi}$ is the twofold covering of the proper Poincare group. As a result the stability subgroup in $\mathbf{\Pi}$ for the standard vector $\overset{o}{p}$ is also the twofold covering of the corresponding stability subgroup in $\mathbf{\Pi}_+^{\uparrow}$. For the representations of the first class (see formula (51)) the appearance of the covering group $\widetilde{\mathbf{E}}(2)$ means that the small cogroup becomes nontrivial and equal to Z_2.

In the situation where $r > 0$ the representations $B^t_{H\uparrow\widetilde{\mathbf{E}}(2)}$ are infinite dimensional. Consequently the induced representation $B^t_{\widetilde{\mathbf{E}}(2)\uparrow\Pi}$ is to be realized in the space of functions whose values are infinite dimensional spinors. We shall not consider here such rather an exotic object.

Let us pass to the case $r = 0$. Here when passing to the small group $\widetilde{\mathbf{E}}(2)$ the small group must be also covered. We denote it by $\widetilde{U}(1)$. The representation parameters in (52)) and (53)) must also change their variation domains:

$$\begin{aligned} D^{\lambda}_{\widetilde{U}(1)}(u_\alpha) &= \exp i\lambda\alpha \qquad (\alpha \in [0, 4\pi), \quad \lambda = 0, \pm 1/2, \pm 1, \pm 3/2, ...), \\ B^{\lambda}_{\widetilde{\mathbf{E}}(2)}(\tau_{\mathbf{b}} u_\alpha) &= D^{\lambda}_{\widetilde{U}(1)}(u_\alpha) = \exp i\lambda\alpha. \end{aligned} \tag{76}$$

Consider the representation

$$B_H^{\lambda,0}(\tau_a h_r) = B_H^{\lambda,0}(\tau_a \tau_{\mathbf{b}} u_\alpha) = D_{\mathbf{P}}^{\overset{o}{p}}(\tau_a) D^{\lambda}_{\widetilde{U}(1)}(u_\alpha) = \exp i\left[\left(\overset{o}{p}a\right) + \lambda\alpha\right] \tag{77}$$

and induce the representation of the Poincare group:

$$\begin{aligned} B^{\lambda,0}_{H\uparrow\Pi}(\Pi)\,\varphi(p) &= B^{\lambda,0}_{H}\left(\Pi_p^{-1}\Pi\Pi_{(\Pi^{-1}p)}\right)\varphi\left(\Pi^{-1}p\right) = \\ &= B^{\lambda,0}\left(\tau_{\left(\Lambda_p^{-1}a\right)}r\left(L,L^{-1}p\right)\right)\varphi\left(L^{-1}p\right) = \\ &= D^{\overset{o}{p}}_{\mathbf{P}}\left(\tau_{\left(\Lambda_p^{-1}a\right)}\right)B^{\lambda}_{\widetilde{\mathbf{E}}(2)}\left(L_p^{-1}LL_{(L^{-1}p)}\right)\varphi\left(L^{-1}p\right) = \\ &= \exp i\,(pa)\,B^{\lambda}_{\widetilde{\mathbf{E}}(2)}\left(L_p^{-1}LL_{(L^{-1}p)}\right)\varphi\left(L^{-1}p\right). \end{aligned} \tag{78}$$

Here functions φ are defined on the space $X \approx \mathbf{\Pi}/H \approx \mathbf{L}/\widetilde{\mathbf{E}}(2) \approx \mathcal{O}_0^{\dot{+}}$. The boost operators L_p are defined by the relation $L_p\left(\sigma\overset{o}{p}\right)L_p^{\dagger} = (\sigma p)$. When the functions of the representation space are transformed by the operators corresponding (in the group $SU(2)$) to rotations round the third basic vector the state function in the standard momentum frame are transformed as an object having the spirality λ:

$$\begin{aligned} B^{\lambda,0}_{H\uparrow\Pi}(r_3(\alpha))\,\varphi\left(\overset{o}{p}\right) &= B^{\lambda}_{\widetilde{\mathbf{E}}(2)}(r_3(\alpha))\,\varphi\left(\overset{o}{p}\right) = \\ &= D^{\lambda}_{\widetilde{U}(1)}(r_3(\alpha))\,\varphi\left(\overset{o}{p}\right) = \exp(i\lambda\alpha)\,\varphi\left(\overset{o}{p}\right). \end{aligned} \tag{79}$$

this representation $B^{\lambda,0}_{H\uparrow\Pi}$ is unitary with respect to the scalar product defined by the expression

$$\begin{aligned} (\varphi_1,\varphi_2)_F &= \int_P d^4p\,\delta\left(p^2\right)\left(\varphi_1(p),\varphi_2(p)\right)_{V^{\lambda}} = \\ &= \int \frac{d^3p}{2\sqrt{p^2}}\varphi_1^{\dagger}(p)\,\varphi_2(p)_{|p=\left(+\sqrt{p^2},p^1,p^2,p^3\right)}. \end{aligned} \tag{80}$$

The properties characteristic to the representations with $m=0$ become essential when the spinor basis is to be formulated. Such representations are necessary. For example we need them to construct representations of the Poincare group containing reflections. The representation $B^{\lambda}_{\widetilde{\mathbf{E}}(2)}$ has a non-trivial kernel. At the same time the simple group $\mathbf{L}$ has no representations whose $\widetilde{\mathbf{E}}(2)$ -restrictions coincide with $B_{\widetilde{\mathbf{E}}(2)}$. In this situation we sacrify the unitarity of the representation. Let us construct an auxiliary representation $C\left(\widetilde{\mathbf{E}}(2)\right)$ with the following properties:

1. it must be exact,

2. its space must contain the subspace of the unitary representation $B^{\lambda}_{\widetilde{\mathbf{E}}(2)}$,

3. on its space must be sufficient to define on it a representation D (an irreducible if possible) of the group $\mathbf{L}$ and the restriction $D_{L\downarrow\widetilde{\mathbf{E}}(2)}$ must be equivalent to $C\left(\widetilde{\mathbf{E}}(2)\right)$.

These conditions fix the practically unique $(2\lambda+1)$-dimensional representation $C^{\lambda}\left(\widetilde{\mathbf{E}}(2)\right)$. Now in the representation

$$\begin{aligned} C^{\lambda,0}_{H\uparrow\Pi}(\Pi) \quad \Phi(p) &= \exp i(pa)\, C^{\lambda}\left(\widetilde{\mathbf{E}}(2)\right)\left(L_p^{-1} L L_{(L^{-1}p)}\right)\Phi\left(L^{-1}p\right) = \\ = \qquad & \exp i(pa)\, D^{\lambda,0}_{L}\left(L_p^{-1}\right) D^{\lambda,0}_{L}(L)\, D^{\lambda,0}_{L}\left(L_{(L^{-1}p)}\right)\Phi\left(L^{-1}p\right) \end{aligned} \tag{81}$$

the spinor basis is introduced in a standard manner:

$$\psi(p) \equiv D^{\lambda,0}_{L}(L_p)\,\Phi(p), \tag{82}$$

$$C^{\lambda,0}_{H\uparrow\Pi}(\Pi)\,\Phi(p) = \exp i(pa)\, D^{\lambda,0}_{L}(L)\,\varphi\left(L^{-1}p\right). \tag{83}$$

Do not forget that this construction is nonunitary.

The procedure exposed above for particles with spirality λ, zero mass and positive energy can be also successfully used for those having negative energy (corresponding to the orbit $\mathcal{O}_0^-$).

6 *Relativistic equations of motion.*

The elementary particles theory has as its main task the construction of models of interacting particles. As far as the interactions are considered to be local (at least for a large part of processes) the particle states are to be fixed in the points of the Minkowski space. To pass from the momentum description to the coordinate one we use the Fourier transform. Consider the general Poincare group $\mathbf{\Pi}_+^\uparrow \approx \mathbf{\Lambda} \rhd \mathbf{P}$ where $\mathbf{\Lambda}$ is the general Lorentz group containing reflections (the subgroup $W = \left\{\widehat{P}, \widehat{T}, \widehat{S}, \widehat{I}\right\}$) and the normal subgroup $\mathbf{\Lambda}_+^\uparrow$. In the previous subsection we have constructed some (in the general setting) reducible representation $B\left(\widetilde{\mathbf{\Pi}}\right)$. The three-dimensional Fourier transform applied to the representation spaces $\mathrm{Fun}\left(\mathcal{O}_m^+, V^{(J,0)} \oplus V^{(0,J)}\right)$ and $\mathrm{Fun}\left(\mathcal{O}_m^+, V^{(J,J)}\right)$ give the coordinate representation in the theory where the time dependance is fixed by the Heisenberg equation and for a free particle is described by the multiplication operator $\exp ip_0 t, \quad p_0 = \sqrt{\mathbf{p}^2 + m^2}$. The creation annihilation processes must be local also with respect to time coordinate t. To make the Fourier transform possible for all the coordinates we are to inject the space $\mathrm{Fun}\left(\mathcal{O}_m^+, V\right)$ into $\mathrm{Fun}\left(P, V\right)$. The elements of the space $\mathrm{Fun}\left(P, V\right)$ are usually called the *wave functions.*

Suppose the projector $\pi\left(m, J, +\right)$ is constructed so that in the space $\mathrm{Fun}\left(P, V\right)$ an irreducible subspace is extracted with respect to the general Poincare transformations.

$$\pi\left(m, J, +\right)\psi\left(p\right) = \psi\left(p\right). \tag{84}$$

The projector $\pi\left(m, J, +\right)$ in the coordinate representation depends on the operator $\widehat{p_\mu} = -i\partial/\partial x^\mu$:

$$\pi_x\left(m, J, +\right)\psi\left(x\right) = \psi\left(x\right). \tag{85}$$

The relations (85) and (85) are called the *relativistic wave equations* for a free relativistic particle in the momentum and coordinate form respectively [28].

Studying the spaces $\mathrm{Fun}\left(\mathcal{O}_m^+, V\right)$ and $\mathrm{Fun}\left(P, V\right)$ we come to the conclusion that the solutions of the equations (85) and (85)describe the particle states with the mass m, spin J, positive energy, fixed P-parity and the reflection phase factorω_S.

Notice that there exist two different reasons for the reducibility of the space $\mathrm{Fun}\,(P,V)$: the little group representation space $V^J \approx V^{(J,0)}$ was changed for $V^{(J,0)} \oplus V^{(0,J)}$ or $V^{(J,J)}$ and also functions defined on the orbit $\mathcal{O}_m^+$ were substituted by the functions on the space P. The projector π must eliminate both these reasons of reducibility. The difference between these reasons will be vivid when the particles with zero spin will be compared with those having $J \neq 0$.

The representation $B_{\widetilde{\mathbf{\Pi}}}^{0m}$ is irreducible on the space $\mathrm{Fun}\left(\mathcal{O}_m^+, V^{(0,0)}\right)$. The projector extracting this subspace from $\mathrm{Fun}\left(\mathcal{O}_m^+, V^{(0,0)}\right)$ evidently has the following structure:

$$\begin{aligned} \left(p^2/m^2\right)\psi\left(p\right) &= \psi\left(p\right) \\ \left(p^2 - m^2\right)\psi\left(p\right) &= 0. \end{aligned} \tag{86}$$

In the coordinate presentation this is just the *Klein-Fock equation*

Consider $B_{\widetilde{\mathbf{\Pi}}}^{m}$ in the space $\mathrm{Fun}\left(\mathcal{O}_m^+, V^{(J,0)} \oplus V^{(0,J)}\right)$. From the explicit realization of the operator $B\left(\widehat{P}\right)$ it follows that the subspace spanned by $\left(\psi\left(p\right), \psi\left(\widehat{P}p\right)\right)$ is invariant with respect to space reflections. This space describes the particle with the spin J as far as in the rest frame the state function transforms under the action of $SU(2)$ group according to the irreducible representation D^J. Thus the necessary projector π in the rest frame coincides with the one isolating the subspace V^J of the subrepresentation $D^J\left(SU(2)\right)$.

It is not difficult to find an operator π in an arbitrary reference system. Let

$$\pi\psi\left(\overset{o}{p}\right) = \psi\left(\overset{o}{p}\right).$$

Then we have

$$\begin{aligned} \pi B\left(L^{-1}\right)B\left(L\right)\psi\left(\overset{o}{p}\right) &= \psi\left(\overset{o}{p}\right), \\ B\left(L\right)\pi B\left(L^{-1}\right)D\left(L\right)\psi\left(L^{-1}\overset{o}{p}\right) &= D\left(L\right)\psi\left(L^{-1}\overset{o}{p}\right), \\ D\left(L\right)\pi D^{-1}\left(L\right)\psi'\left(p\right) &= \psi'\left(p\right), \end{aligned}$$

with $p \equiv L^{-1}p$. We shall use the notation $\pi\left(p\right) \equiv D\left(L\right)\pi D^{-1}\left(L\right)$ and the standard decomposition $L = L_p L_R$. The projector π intertwine the representations of $SU(2)$, that is π commutes with $D\left(L_R\right)$. Thus,

$$\pi\left(p\right) = D\left(L_p\right)\pi D^{-1}\left(L_p\right). \tag{87}$$

Exercise 6 - 1 *Show that the operator $\pi(p)$ is covariant:*

$$\pi(p) \xrightarrow{L} \pi(Lp).$$

Consider now the space $\mathrm{Fun}\left(P, V^{(J,0)} \oplus V^{(0,J)}\right)$. The operator π must not only subtract the irreducible subspace V^J but also put the momentum of the particle in the fixed orbit $\mathcal{O}_m^+$. For the parameter p in (87) this condition is fulfilled (provided $\left(\overset{o}{p} \in \mathcal{O}_m^+\right)$). The function $\psi(p)$ is the solution of the equation

$$\pi(p)\,\psi(p) = \psi(p), \tag{88}$$

only when its argument is in the orbit $\mathcal{O}_m^+$. Finally the set of solutions to (88) constitute the space of the irreducible representation $B^{Jm+}_{\eta_{\tilde{P}}\omega_S}\left(\widetilde{\mathbf{\Pi}}\right)$. The equation (88) is the general relativistic covariant equation of motion for massive particles with spin. The specific form of the projector $\pi(p)$ depends on what space $\mathrm{Fun}(P, V)$ is used as a comprehending space.

Now we shall consider the construction of the Dirac equation. The corresponding irreducible representation is $B^{m,1/2}\left(\widetilde{\mathbf{\Pi}}\right)$. Consider the space of wave functions $\mathrm{Fun}\left(P, V^{(1/2,0)} \oplus V^{(0,1/2)}\right)$. To extract the spin $J = 1/2$ particles it is sufficient to project on one of the subspaces in the direct sum $V^{(1/2,0)} \oplus V^{(0,1/2)}$:

$$\pi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\gamma_0 + I). \tag{89}$$

Here γ_0 is diagonal $\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Perform a substitution $\gamma_0 = \gamma_\mu \left(\overset{o}{p}\right)^\mu m^{-1}$. Using (87) we find $\pi(p)$:

$$\begin{aligned} \pi(p) &= D(L_p)\,\pi D\left(L_p^{-1}\right) = \\ &= \frac{1}{2m}\left(D^{(1/2,0)\oplus(0,1/2)}(L_p) \times \gamma_\mu \left(\overset{o}{p}\right)^\mu D^{(1/2,0)\oplus(0,1/2)}\left(L_p^{-1}\right) + m\right) = \\ &= \frac{1}{2m}\left(\gamma_\mu p^\mu + m\right). \end{aligned} \tag{90}$$

Introducing this expression in the equation (88) we get the *Dirac equation*

$$\left(\gamma_\mu p^\mu - m\right)\psi(p) = 0. \tag{91}$$

Here each component of the solution $\psi(p)$ is subject to the Klein-Fock equation. The latter is a result of the condition $(\pi(p))^2 = \pi(p)$ that is equivalent to the relation $p^2 = m^2$.

Consider the scalar product in the space of Dirac wave functions. Here the operators $D^{(1/2,0)\oplus(0,1/2)}(L)$ are block diagonal, the space reflection operator coincides with the matrix $\eta_{\widehat{P}}\gamma_0$:

$$(\psi_1, \psi_2)_F = \int \frac{d^3p}{2p_0} \psi_1^\dagger(p)\, \gamma_0 \psi_2(p) = \int \frac{d^3p}{2p_0} \overline{\psi_1}(p)\, \psi_2(p)\,, \tag{92}$$

In the above expression the Dirac conjugation $\psi(p) \to \overline{\psi(p)} \equiv \psi^\dagger(p)\,\gamma_0$ was introduced.

When studying the elementary particles interactions the second quantization procedure must be performed and we are to pass to quantum field theoretical description. There the wave functions considered above must be changed for the corresponding field operators. The latter still obey the free equations of motion when considered in the so called interaction representation.

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