7 Evolution equations

At large energies $\sqrt{s}$ an effective parameter of the perturbation theory apart from α_{em} in QED or α_s in QCD can contain an additional large factor being a certain power of $\ln s$. For example, in QED the effective parameter is $\alpha_{em} \ln^2 s$ for the electron Sudakov form-factor [1] and for the amplitude of the backward $e\overline{e}$-scattering [2]. The corresponding physical quantities in the double-logarithmic approximation (DLA) are obtained by calculating and summing the asymptotic contributions $\sim \alpha_{em}^n \ln^{2n} s$ in all orders of the perturbation theory. Their region of applicability is

$$\alpha_{em} \ll 1\,,\ \alpha_{em} \ln^2 s \sim 1\,. \tag{1}$$

In DLA the transverse momenta $|k_r^\perp|$ of the virtual and real particles are large and are implied to be strongly ordered because the first logarithm $\ln s$ in each one-loop diagram is obtained as a result of integration over the energy of a relatively soft particle and another logarithm - over its transverse momentum (or emission angle). Instead of calculating each individual diagram one can initially divide the integration region in several subregions in dependence from the ordering of the particle transverse momenta and to sum subsequently the contributions with the same orderings over all diagrams in the given order of perturbation theory. It gives a possibility to write an evolution equation with respect to the logarithm of the infrared cut-off λ [3]. Below we consider a simple example of such equation for the case of the radiative corrections to the Z-boson production in the $e\overline{e}$ collisions. In the Regge kinematics in some cases the integrals over transverse momenta are convergent, which leads to the effective parameter of the perturbation theory $\alpha \ln s$ [4] or even $\alpha^2 \ln s$ [5].

For the inclusive processes at large energies and large momentum transfers Q there is a strong ordering only over transverse momenta of particles. In this case one can use the probabalistic interpretation to express the hadron-lepton processes it terms of the cross-sections for the collision of the point-like objects - partons. We review below the parton model (see also [6]) and discuss its modification in QCD, where the parton transverse momenta slowly grow with increasing Q. Later we derive the evolution equations for parton distributions and for fragmentation functions, calculate the splitting kernels and find the solutions. The relation to the renormalization group in the framework of the Wilson operator product expansion is also discussed. The evolution equations are generalized for the parton correlators describing matrix elements of the so-called quasi-partonic operators. In more details we study evolution equations for the twist-3 quasi-partonic operators describing the large-Q^2 behaviour of the structure function $g_2(x)$. Next-to-leading corrections to the splitting kernels are reviewed.

7.1 Sudakov resummation of double-logarithmic terms in QED

As an example of the necessity to go beyond the usual perturbation theory with the use of evolution equations in the parameter of the infrared cut-off we consider here the Z boson production in the $e\overline{e}$ colliding beams. The corresponding total cross-section according to the optical theorem is proportional to the s-channel discontinuity of the forward scattering amplitude (see Fig. 1). At large energies $s = (p_e + p_{\overline{e}})^2 \gg m_e^2$ we have

$$\sigma_{tot} = \frac{1}{2s}\frac{1}{i}\,\Delta\, A(s,0)\,,\ \Delta\, A(s,0) = A(s+i\epsilon,0) - A(s-i\epsilon,0)\ . \tag{2}$$

Using the Feynman rules for the Standard Model we can write the elastic amplitude for high energy e^-e^+ forward scattering with the virtual Z-boson in the s channel as follows

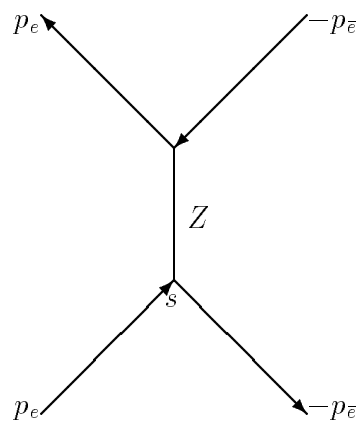

Figure 1: Feynman graph for the $e\bar{e}$ forward scattering through the Z-production

$$A(s,0) = -\left(\frac{g}{2\cos\theta_w}\right)^2 \frac{\frac{1}{4}Tr\left(\gamma_\sigma(g_v+g_a\gamma_5)\widehat{p_e}\,\gamma_\sigma\,(g_v+g_a\gamma_5)\,\widehat{p_{\bar{e}}}\right)}{s-M_Z^2+i\,\Gamma_{tot}\;M_Z}\,, \tag{3}$$

where g is the weak coupling constant, θ_w is the Weinberg angle, M_Z and Γ_{tot} are the mass and total width of the Z-boson. The parameters g_v and g_a are the effective vector and axial couplings for the electron

$$g_v = -\frac{1}{2} + 2\sin^2\theta_w\,,\;\; g_a = -\frac{1}{2}\,. \tag{4}$$

In eq. (3) we neglected the electron mass and performed an averaging over the initial particle spins using the projectors $\widehat{p_e}$ and $-\widehat{p_{\bar{e}}}$ to their physical states.

As a result, the total cross-section (2) near the resonance ($|s-M_Z^2| \sim \Gamma_{tot}\;M_Z$) is given by the Breit-Wigner expression

$$\sigma_Z(s) = (g_v^2+g_a^2)\left(\frac{g}{2\cos\theta_w}\right)^2 \frac{\Gamma_{tot}}{4\,M_Z\,(E_{nr}^2+\frac{1}{4}\Gamma_{tot}^2)}\,, \tag{5}$$

where $E_{nr} = W - M_Z$ ($W = \sqrt{s}$) is the non-relativistic energy of the Z-boson. We can rewrite the cross-section in the simple form

$$\sigma_Z(s) = \frac{3\pi}{M_Z^2}\;\frac{\Gamma_{e\bar{e}}\,\Gamma_{tot}}{E_{nr}^2+\frac{1}{4}\Gamma_{tot}^2} \tag{6}$$

with the use of the following expression for the $e\bar{e}$ partial width of the Z -boson

$$\Gamma_{e\bar{e}} = g^2\,\frac{(g_v^2+g_a^2)}{48\,\pi}\frac{M_z}{\cos^2\theta_w}\,. \tag{7}$$

It turns out, that the radiative corrections in the powers of the electromagnetic fine structure constant $\alpha_e = e^2/(4\pi) \simeq 1/137$ are essential for the Z production in $e\bar{e}$ collisions. They diminish significantly the total cross-section in the maximum at $E_{nr} = 0$ and change

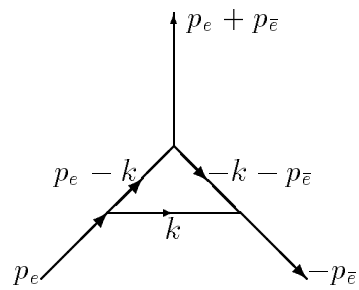

Figure 2: One loop diagram for the Sudakov vertex

the form of its s-dependence. Moreover, the effective parameter of the perturbation theory in this case turns out to be

$$\frac{\alpha_e}{\pi}\,\ln^2\frac{M_Z^2}{m_e^2} \simeq 1.331 \tag{8}$$

and therefore we should sum the contributions of all orders of perturbation theory in the double-logarithmic approximation (DLA) [1] valid in the region

$$\frac{\alpha_e}{\pi}\,\ln^2\frac{M_Z^2}{m_e^2} \sim 1\,,\;\; \alpha_e \ll 1\,. \tag{9}$$

Note, that for the small energy resolution $\epsilon \ll m_e$ the argument of one of logarithms should be substituted by M_Z^2/ϵ^2 (see eq. 2.11.2 in ref.[6]).

To begin with, let us consider the one loop correction to the electromagnetic vertex at high photon virtuality $s \gg m_e^2$, where m_e is the mass of the electron (see Fig. 2). We use the Sudakov variables for the momentum k of the virtual photon with the non-zero mass λ (introduced for the infrared regularization) [1]:

$$k = \alpha\, p_e' + \beta\, p_{\bar{e}}' + k_\perp\,,\;\; (k_\perp, p_e) = (k_\perp, p_{\bar{e}}) = 0\,,\;\; d^4k = \frac{s'}{2}\, d\,\alpha\, d\,\beta\, d^2k_\perp\,. \tag{10}$$

Here p_e' and $p_{\bar{e}}'$ are light-cone momenta ($p_i'^2 = 0$) constructed as linear combinations of momenta p_e and $p_{\bar{e}}$ of the colliding electron and positron:

$$p_e = p_e' + \frac{m_e^2}{s'} p_{\bar{e}}'\,,\;\; p_{\bar{e}} = p_{\bar{e}}' + \frac{m_e^2}{s'} p_e'\;,\; s' = 2p_e'\, p_{\bar{e}}' \simeq s\,. \tag{11}$$

The Sudakov variable α (β) has the physical meaning of the the photon energy measured in units of the electron (positron) energy in its infinite momentum frame [8].

Taking into account, that in the essential region of integration the Sudakov parameters are small

$$|\alpha| \ll 1\,,\;\; |\beta| \ll 1\,,\;\; \vec{k_\perp}^{\,2} = -k_\perp^2 \ll \sqrt{s}\,, \tag{12}$$

we can simplify the spin structure in the matrix element for the vector (or pseudo-vector) vertex

$$\overline{v}(-p_{\bar{e}})\,\gamma_\sigma\,(-\widehat{p}_{\bar{e}} - \widehat{k} + m_e)\,\gamma_\mu\,(\widehat{p}_e - \widehat{k} + m_e)\,\gamma_\sigma\, u(p_e) \simeq -2s\,\overline{v}(-p_{\bar{e}})\,\gamma_\mu\; u(p_e)\,, \tag{13}$$

where $u(p_e)$ and $\overline{v}(-p_{\overline{e}})$ are the spinors describing the initial electron and positron, respectively. As a result, the relative correction to the vertex can be written in the Sudakov variables as follows

$$\Delta\gamma \simeq$$

$$\frac{is^2\,e^2}{(2\pi)^4}\int \frac{d\,\alpha\,d\,\beta\,d^2\overrightarrow{k_\perp}}{s\alpha\beta-\overrightarrow{k_\perp}^2-\lambda^2+i\varepsilon}\,\frac{1}{s\alpha(\beta-1)-\overrightarrow{k_\perp}^2-m_e^2\beta+i\varepsilon}\,\frac{1}{s\beta(\alpha+1)-\overrightarrow{k_\perp}^2+m_e^2\alpha+i\varepsilon}\,. \tag{14}$$

The integral over the variable α can be calculated by residues. It is non-zero only if β belongs to the interval $-m_e^2/s<\beta<1$. Moreover, the essential region of the integration over β is $m_e^2/s\ll\beta\ll 1$. Here by taking the residue in the pole at

$$\alpha=-\frac{\overrightarrow{k_\perp}^2+m_e^2\beta}{(1-\beta)\,s}$$

and averaging over the directions of the vector $\overrightarrow{k_\perp}$we can obtain

$$\Delta\gamma\simeq-\frac{e^2}{8\pi^2}\int_{m_e^2/s}^1 d\beta\int_0^\infty d(|k_\perp|^2)\frac{1}{|k_\perp|^2+m_e^2\beta^2+\lambda^2}\,\frac{s}{s\beta-|k_\perp|^2+m_e^2\alpha+i\varepsilon}\,. \tag{15}$$

The essential region of integration over $|k_\perp|^2$ is $\max(\lambda^2,m_e^2\beta^2)\ll|k_\perp|^2\ll s\beta$. We return to the initial Sudakov variables with the use of the relation $\alpha\equiv|k_\perp|^2/(s\beta)$ to obtain a more symmetric expression valid with the same accuracy

$$\Delta\gamma\simeq-\frac{e^2}{8\pi^2}\int_0^1\int_0^1\frac{d\,\alpha\,d\beta\;\theta(s\alpha\beta-\lambda^2)}{\left(\alpha+\frac{m_e^2}{s}\beta\right)\left(\beta+\frac{m_e^2}{s}\alpha\right)}\,. \tag{16}$$

In the new variables the essential region of integration is

$$\frac{m_e^2}{s}\beta\ll\alpha\ll 1\,,\;\;\frac{m_e^2}{s}\alpha\ll\beta\ll 1\,,\;\;s\alpha\beta\gg\lambda^2. \tag{17}$$

Note, that the result (16) can be obtained directly from expression (14) by integrating initially over $\vec{k}_\perp^2$ with the use of the substitution

$$\frac{1}{s\alpha\beta-\overrightarrow{k_\perp}^2-\lambda^2+i\varepsilon}\to-i\pi\,\theta(s\alpha\beta-\lambda^2)\;\delta(s\alpha\beta-\overrightarrow{k_\perp}^2)\,, \tag{18}$$

corresponding to the neglection of the small contribution from the integral with the principal value prescription. Such procedure means, that in the essential region of integration (17) the photon with the momentum k lies on its mass shell.

In expression (16) one can introduce the logarithmic variables

$$\xi=\ln\frac{s\,\alpha}{m_e^2}\,,\;\;\eta=\ln\frac{s\,\beta}{m_e^2}\,,\;\;\rho=\ln\frac{s}{m_e^2}\,,\;L=\ln\frac{m_e^2}{\lambda^2} \tag{19}$$

to obtain its value with the double-logarithmic accuracy (see Fig. 3 for the integration region)

$$\Delta\gamma\simeq$$

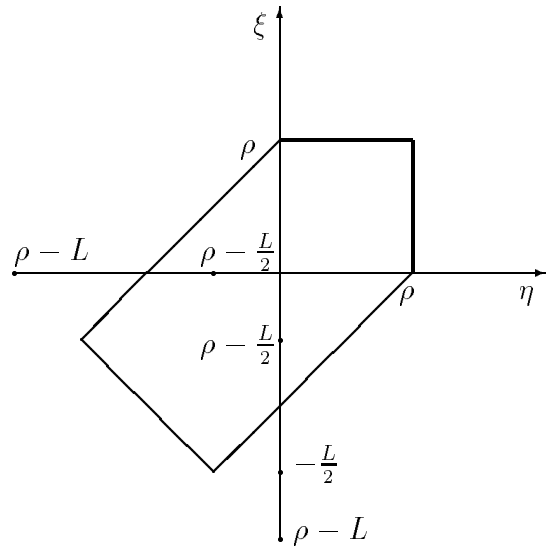

Figure 3: DL region of integration in the Sudakov vertex

$$-\frac{e^2}{8\pi^2}\int_{-\infty}^{\rho} d\xi \int_{-\infty}^{\rho} d\eta\, \theta(\xi+\eta-\rho+L)\,\theta(\xi-\eta+\rho)\,\theta(\eta-\xi+\rho) = -\frac{\alpha_{em}\ln\frac{s}{m_e^2}}{2\pi}\left(\frac{\ln\frac{s}{m_e^2}}{2}+\ln\frac{m_e^2}{\lambda^2}\right). \tag{20}$$

The virtual photon momentum is collinear to momenta p_e ($p_{\overline{e}}$) for $\beta \gg \alpha$ ($\alpha \gg \beta$). Therefore we can write the result for the Sudakov vertex in one loop in another form

$$\Delta\gamma = -2\frac{\alpha_{em}}{\pi}\int_{\lambda}^{\sqrt{s}}\frac{d\omega}{\omega}\int_{max(\frac{m}{\sqrt{s}},\frac{\lambda}{\omega})}^{1}\frac{d\theta}{\theta} = -2\frac{\alpha}{\pi}\left(\ln\frac{m}{\lambda}\ln\frac{\sqrt{s}}{m}+\frac{1}{2}\ln^2\frac{\sqrt{s}}{m}\right), \tag{21}$$

where $\omega = \beta\sqrt{s}$ (for $\beta > \alpha$) or $\omega = \alpha\sqrt{s}$ (for $\alpha > \beta$) and $\theta = |k_\perp|/\omega$ are the photon frequency and emission angle.

The final result for the Sudakov vertex in the leading logarithmic approximation is [1]

$$\gamma = e^{\Delta\gamma} \tag{22}$$

and can be obtained with the use of the evolution equation in the photon mass λ [3]. It plays a role of the infrared cut-off in the transverse components $|k_{i\perp}|$ of momenta of virtual photons:

$$|k_{i\perp}| \gg \lambda\,. \tag{23}$$

To prove (22) we extract from all Feynman diagrams for the electromagnetic vertex Γ_μ the line for the photon having the minimal transverse momentum $|k_\perp|$ such that for other photons i the value of $|k_\perp|$ plays role of λ

$$|k_{i\perp}| \gg |k_\perp| \gg \lambda\,. \tag{24}$$

According to V.N. Gribov [7] the integration over momenta of such soft photons is factorized from the contributions with larger photon transverse momenta. For example, in the region

$\lambda \ll m$ the evolution equation for γ in the variable $\ln(m/\lambda)$ has the form

$$\frac{\partial}{\partial \ln \frac{m}{\lambda}} \gamma = \gamma \, \frac{2\,\alpha_{em}}{\pi} \, \ln \frac{\sqrt{s}}{m} \tag{25}$$

with the initial condition

$$\gamma_{|\lambda=m} = \exp\left(-\frac{\alpha_{em}}{\pi} \, \ln^2 \frac{\sqrt{s}}{m} \right) . \tag{26}$$

The solution of the this equation is given by the Sudakov expression (22).

The Gribov arguments are based on the investigation of the amplitude, obtained from the Sudakov vertex by extracting the virtual photon with the minimal transverse momentum $k_\perp$ [7]. One can write for it the double dispersion relation in the variables $(p_e - k)^2$ and $(p_{\bar{e}} + k)^2$. The pole contributions in these invariants corresponds to the factorization of the softest photon and leads to the above evolution equation. On the other hand, the contributions from the cuts are negligible, because the photon production amplitudes in the collision of the virtual soft photon with the electron or positron are small $\sim |k_\perp|$ due to the fact, that such photon interacts with the total electric charge and therefore its matrix elements between different states are zero [7].

In the case of the Z-boson production in the region $W - M \ll M$ ($W = \sqrt{s}$) apart from the suppression due to the virtual photon corrections proportional to γ^2 we should take into account in the total cross-section σ_t also the photon bremsstrahlung contributions cancelling the infrared divergences at $\lambda \to 0$

$$\sigma_t =$$

$$e^{2\,\Delta\gamma} \sum_{n=0}^{\infty} \left(4 \frac{\alpha_{em}}{\pi} \right)^n \frac{1}{n!} \prod_{r=1}^{n} \left(\int_\lambda^W \frac{d\omega_r}{\omega_r} \int_{max(\frac{m}{W}, \frac{\lambda}{\omega})}^1 \frac{d\theta_r}{\theta_r} \right) \int_0^\infty d\omega \; \delta(\omega - \sum_{l=1}^{n} \omega_l) \, \sigma_Z((W-\omega)^2) \; . \tag{27}$$

Here we introduced the photon total energy $\omega = \sum_{l=1}^{n} \omega_l$.

Expression (27) leads to an increase of the cross-section in the energy interval $M_Z \gg W - M_Z \gg \Gamma_Z$ because the emission of photons with the frequency $\omega \simeq W - M_Z$ can return the initial particles $e^\pm$ to the resonance region where the cross-section is large $\sigma(W^2) \sim (\alpha_{em}\, \Gamma_{e\bar{e}}/M_Z^2)(W - M_Z)^{-1}$. This effect results in an appearance of the radiative tail for the total cross-section σ_t at $W > M_Z$. In the particular case when the produced photon energy is small $W - M_Z \ll M_Z$ with the use of the representation

$$2\,\pi\, \delta(\omega - \sum_{l=1}^{n} \omega_l) = \int_{-\infty}^{\infty} d\,t \; e^{it\,(\omega - \sum_{l=1}^{n} \omega_l)} \tag{28}$$

one can write the cross-section for the inclusive production of the Z -boson in a simple form (taking into account, that $\omega_l \ll 1/t$)

$$\sigma_t \simeq \int_0^{M_Z} \sigma_Z((W-\omega)^2)\, d\omega \int_{-\infty}^{\infty} \frac{d\,t}{2\pi} \, e^{it\,\omega} \left(\min\left(\frac{1}{|t| M_Z}, \, 1 \right) \right)^{\frac{4\alpha}{\pi} \ln \frac{W}{m_e}} . \tag{29}$$

Thus, effectively the cross-section is expressed in terms of the squared Sudakov form-factor (22) in which instead of the condition $|k_\perp| > \lambda$ for the infrared cut-off of the virtual

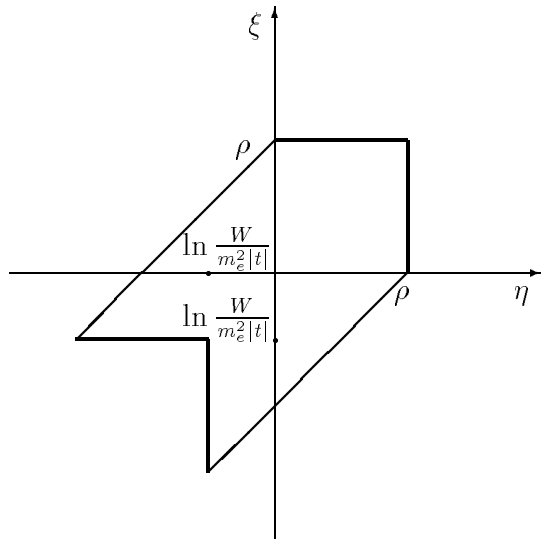

Figure 4: DL region of integration in the Z-production cross-section

photon momenta one should use the constraint $\omega = W(\alpha+\beta) > 1/|t|$ (see Fig. 4). The expression (29) can be written in another form

$$\sigma_t = \int_0^{M_Z} \sigma_Z((W-\omega)^2)\, d\omega\, \frac{d}{d\omega}\left(\frac{\omega}{M_Z}\right)^{\frac{4\alpha_{em}}{\pi}\ln\frac{W}{m_e}}$$

$$= \int_0^{M_Z} \sigma((W-\omega)^2)\, \frac{d\omega}{\omega}\, \frac{4\alpha_{em}}{\pi}\, \ln\frac{W}{m_e}\left(\frac{\omega}{M_Z}\right)^{\frac{4\alpha_{em}}{\pi}\ln\frac{W}{m_e}}. \tag{30}$$

Here $\omega \ll M_Z$ is the total energy of the produced photons. From these formulas one can see, that the shape of the distribution in W is changed (leading to a tail at $W > m_Z^2$) but the integral from σ_t over W remains the same due to the Sudakov suppression.

Let us interpret expression (30) in a probabalistic way. Physically it is possible, because in the double-logarithmic approximation the quantum interference effects are small. To begin with, we introduce the momenta $x_1 p_e$ and $x_2 p_{\bar{e}}$ of the bare electron and positron inside the corresponding physical objects. The bare particles are called partons [9, 10]. They are considered as point-like objects. The physical particles with definite masses are composite states of partons (see for example Ref. [6]). The total photon energy is $\omega = W - W(x_1+x_2)/2 \ll W$. The above expression for σ_t can be presented as follows

$$\sigma_t = \int_0^1 dx_1 \int_0^1 dx_2\, \sigma_Z(s x_1 x_2)\, n(x_1)\, n(x_2)\,,\ \ s x_1 x_2 \simeq W^2\,(x_1+x_2-1) \simeq (W-\omega)^2\,, \tag{31}$$

where $n(x_1)$ and $n(x_2)$ are the inclusive densities of the corresponding bare particles inside physical ones. The result (31) corresponds to the physical picture, in which the Z-boson is produced in the collision of the bare electron and positron (see Fig. 5).

The density of the number of bare electrons inside the physical electron for $x \to 1$ is given in DLA by the expression

$$n(x) = \frac{2\alpha_{em}}{\pi}\ln\frac{W}{m_e}\,(1-x)^{-1+\frac{2\alpha_{em}}{\pi}\ln\frac{W}{m_e}}. \tag{32}$$

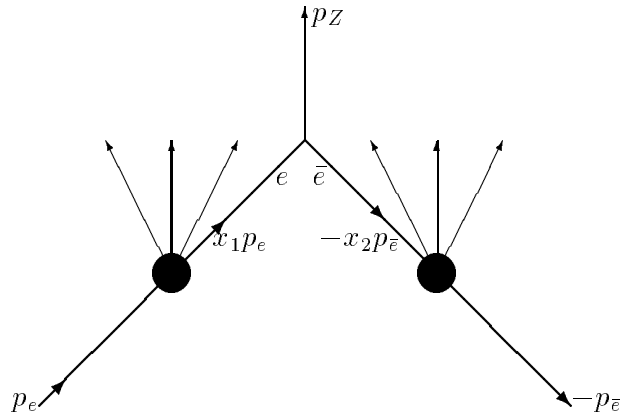

Figure 5: Partonic description of the Z-production in $e\bar{e}$ collisions

The equivalence of two above expressions for σ_t is a consequence of the following relation valid for small $\delta = \frac{2\alpha_{em}}{\pi} \ln \frac{W}{m_e}$

$$\int_0^{\epsilon} d\epsilon_1 \, n(1-\epsilon_1) \, n(1-\epsilon+\epsilon_1) \simeq \frac{4\alpha_{em}}{\pi} \ln \frac{W}{m_e} \, \epsilon^{-1+\frac{4\alpha_{em}}{\pi} \ln \frac{W}{m_e}} \, , \; \epsilon_1 = 1 - x_1 = \frac{2\omega_1}{W} \, , \tag{33}$$

obtained with the use of the equality

$$\delta^2 \int_0^1 d\, z \, z^{-1+\delta} (1-z)^{-1+\delta} = \delta^2 \frac{\Gamma(\delta)\Gamma(\delta)}{\Gamma(2\delta)} \simeq 2\,\delta \, . \tag{34}$$

The density $n(x)$ satisfies the sum rule

$$\int_0^1 d\, x \, n(x) = 1 \, , \tag{35}$$

meaning that in our approximation ($x \to 1$) there is only one bare electron inside of the physical one. Furthermore, we obtain

$$\int_0^1 d\, x \, x \, n(x) = 1 - \delta \, , \tag{36}$$

corresponding to the fact, that in DLA the relative energy ω/E taken by the bare photons from the physical electron is small $\omega/E \simeq \delta$. Nevertheless, the number of the bare photons with a small frequency is infinite, which is important for the high energy processes of the type $e\bar{e} \to e\bar{e} + anything$ described by the method of equivalent photons.

The partonic expression for the cross-section σ_t in terms of the product of $n(x_{1,2})$ is valid also far from the resonance region $W - M_Z \sim M_Z$, but in this case we should sum also single logarithmic terms $\left(\alpha_{em} \ln \frac{W}{m_e}\right)^n$ and $n(x)$ can be found from the evolution equations for partonic distributions in QED (see Ref. [20]).

Similar formulas are valid in the framework of the parton model for the W and Z production in the hadron-hadron collisions but in this case the suppression of the resonance production is related to the contribution of the virtual gluons and quarks.

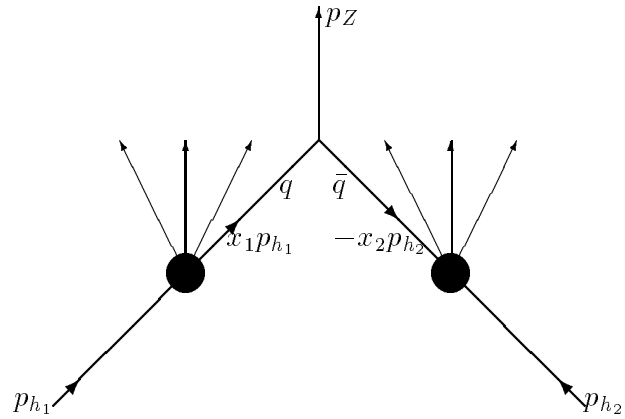

Figure 6: Partonic description of the Z-production in $h_1 h_2$ collisions

7.2 Parton model in QCD

It is known, that hadrons are composite states of the point-like bare particles - partons having quantum numbers of quarks and gluons [9, 10]. For example, in the framework of the parton model the cross-section for the inclusive Z production in hadron-hadron collisions $h + h \to Z + anything$ is expressed in terms of the product of the inclusive probabilities $D^q_{h_1}(x_1)$, $D^{\overline{q}}_{h_2}(x_2)$ to find the quark and anti-quark with the energies $x_{1,2}\sqrt{s/4}$ inside the colliding hadrons h_1, h_2 and the Born cross-section $\sigma_{q\overline{q}\to Z}\,(s\,x_1 x_2)$ (2) for the boson production in the quark-anti-quark collisions [12, 13]. This expression is integrated over the Feynman components $x_{1,2}$ of momenta of the quark and anti-quark (see Fig. 6)

$$\sigma_{h_1 h_2 \to Z} = \frac{1}{3}\sum_q \int dx_1 dx_2 \left(D^q_{h_1}(x_1)\, D^{\overline{q}}_{h_2}(x_2) + D^{\overline{q}}_{h_1}(x_1)\, D^q_{h_2}(x_2)\right) \sigma_{q\overline{q}\to Z}\,(s x_1 x_2)\,. \tag{37}$$

Here the factor $\frac{1}{3}$ appears because in the quark and anti-quark distributions $D^{q,\overline{q}}_h(x)$ the sum over three color states of the quark is implied, but the Z-boson is produced in annihilation of quarks and anti-quarks with an opposite color.

Initially the parton model was applied to the description of the deep-inelastic scattering of leptons off hadrons [9, 10]. For example, the differential cross-section of the inclusive scattering of electrons with the initial and final momenta p_e and p'_e respectively off the hadron with its momentum $p = p_h$ can be written as follows [6] (see Fig. 7)

$$d\sigma_\gamma = \frac{\alpha^2}{\pi}\frac{1}{q^4} L^{\mu\nu} W_{\mu\nu} \frac{d^3 p'_e}{(p p_e) E'}\,,\; E' = |p'_e|\,, \tag{38}$$

where only the exchange of the photon with the momentum $q = p_e - p'_e$ was taken into account (for large q^2 one should also add the Z-boson exchange, which leads in particular to the parity non-conservation effects proportional to $g_a \sim T_3$). The electron tensor $L^{\mu\nu}$ is calculated explicitly

$$L^{\mu\nu} = \frac{1}{2} tr\,(\widehat{p'_e}\gamma_\mu \widehat{p_e}\gamma_\nu) = 2\,(p'^\mu_e p^\nu_e + p'^\nu_e p^\mu_e - \delta^{\nu\mu}(p'_e p_e))\,. \tag{39}$$

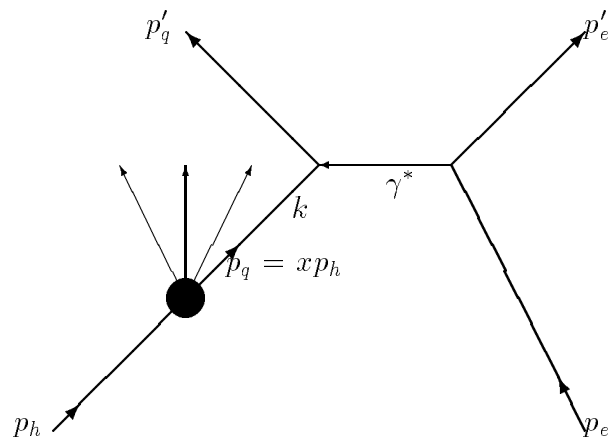

Figure 7: Partonic description of the deep-inelastic eh scattering

The hadronic tensor is expressed in terms of matrix elements of the electromagnetic current J^{el}_μ

$$W_{\mu\nu} = \frac{1}{4}\sum_n \langle p \left| J^{el}_\mu(0) \right| n \rangle \langle n \left| J^{el}_\nu(0) \right| p \rangle (2\pi)^4 \delta^4(p+q-p_n). \tag{40}$$

It can be expressed in terms of two structure functions $F_{1,2}(x,Q^2)$ using the properties of the gauge-invariance and the parity conservation as follows

$$\frac{1}{\pi}W_{\mu\nu} = -\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(x,Q^2) + \left(p_\mu - \frac{q_\mu (pq)}{q^2}\right)\left(p_\nu - \frac{q_\nu (pq)}{q^2}\right)\frac{F_2(x,Q^2)}{pq}. \tag{41}$$

Here Q^2 and x are the Bjorken variables [9]

$$Q^2 = -q^2\,,\ x = \frac{Q^2}{2pq}. \tag{42}$$

The structure functions $F_{1,2}(x,Q^2)$ do not depend on Q^2 for fixed x and large Q^2 in the framework of the Bjorken-Feynman parton model

$$\lim_{Q\to\infty} F_{1,2}(x,Q^2) = F_{1,2}(x)\,, \tag{43}$$

which corresponds to the Bjorken scaling [9]. In this model the structure functions can be calculated in the impulse approximation as a sum of the structure functions for the charged partons averaged with the partonic distributions (see Fig. 7). From the point of view of the Wilson operator product expansion [11] the Bjorken scaling means, that the obtained composite operators have canonical dimensions, as in the free theory.

The charged partons are assumed to be fermions (quarks). The hadronic tensor for the quarks can be calculated with the use of the relation

$$|n\rangle\langle n| = \int \frac{d^4p_n}{(2\pi)^3}\,\delta(p_n^2 - m^2)\,\theta(E_n)\,, \tag{44}$$

in the form

$$\frac{1}{\pi}W^f_{\mu\nu} = \frac{Q_q^{2x}}{2}\delta\left((k+q)^2\right)\frac{1}{2}tr\left(\hat{k}\,\gamma_\mu(\hat{q}+\hat{k})\gamma_\nu\right) = -Q_q^2\frac{1}{2}\,\delta^\perp_{\mu\nu}\,\delta(\beta-x)\,, \tag{45}$$

where β is the Sudakov variable $\beta = \frac{kq}{pq}$ of the quark with its momentum equal to k and $x = \frac{Q^2}{2pq}$ is the Bjorken variable.

With the use of the identity

$$-\delta^\perp_{\mu\nu} = -\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \left(p_\mu - \frac{q_\mu(pq)}{q^2}\right)\left(p_\nu - \frac{q_\nu(pq)}{q^2}\right)\frac{2x}{pq} \tag{46}$$

we obtain, that in the framework of the quark-parton model the Callan-Gross relation between F_1 and F_2 is valid [14]

$$F_2(x) = 2xF_1(x)\,, \tag{47}$$

where the expression

$$F_2(x) = x\sum_{i=q,\overline{q}} e_i^2\, n^i(x) \tag{48}$$

corresponds to the impulse approximation for the cross-section. The quantity $n^q(x)$ is the quark distribution in the hadron, normalized in such way, that the electric charge conservation for the proton takes the form

$$1 = \sum_{i=q,\overline{q}} e_i \int_0^1 dx\, n^i(x)\,. \tag{49}$$

The structure functions can be expressed in terms of the cross-sections σ_t and σ_l for the scattering of the virtual photons off protons with the transverse (t) and longitudinal (l) polarization [6]. In the quark-parton model $\sigma_l = 0$ in an accordance with the Callan-Gross relation between F_1 and F_2.

Another important deep-inelastic process is the scattering of neutrinos and anti-neutrinos off hadrons. The first contribution is induced by the Z-boson exchange (a neutral current contribution). In this case the expressions for inclusive cross-sections are similar to the case of the electron scattering.

The process of the neutrino-hadron scattering with a charged lepton in the final state is related to the t-channel exchange of the W-boson. The cross-section of the corresponding inclusive process, in which one measures only a produced electron or positron with the momentum p'_e, is given below [6]

$$d\sigma^{\nu,\overline{\nu}} = \frac{g^4}{16\pi^3}\left(\frac{1}{2\sqrt{2}}\right)^2\left(\frac{1}{q^2-M_W^2}\right)^2 L^{\mu\rho}_{\nu,\overline{\nu}}W^{\nu,\overline{\nu}}_{\mu\rho}\frac{d^3p'_e}{(pp_{\nu,\overline{\nu}})E'}\,. \tag{50}$$

Taking into account, that the initial neutrino or anti-neutrino have fixed helicities, we obtain

$$L^{\mu\rho}_{\nu,\overline{\nu}} = tr(\widehat{p'_e}\gamma^\mu(1\pm\gamma_5)\widehat{p_{\nu,\widehat{\nu}}}\gamma^\rho(1\pm\gamma_5)) = 8\left(p'^\mu_e p^\rho_{\nu,\overline{\nu}} + p'^\rho_e p^\mu_{\nu,\overline{\nu}} - \delta^{\rho\mu}(p'_e p_{\nu,\overline{\nu}}) \pm i\epsilon^{\mu\rho\lambda\delta}p^\lambda_{\nu,\overline{\nu}}\,p'^\delta_e\right)\,. \tag{51}$$

For the hadronic tensor one has a more general spin structure in comparison with the deep-inelastic electron-hadron scattering where the parity non-conservation effects were absent

$$\frac{1}{\pi}W^{\nu,\overline{\nu}}_{\mu\rho} = -\left(\delta_{\mu\rho} - \frac{q_\mu q_\rho}{q^2}\right)F_1^{\nu,\overline{\nu}} + \left(p_\mu - \frac{q_\mu(pq)}{q^2}\right)\left(p_\rho - \frac{q_\rho(pq)}{q^2}\right)\frac{F_2^{\nu,\overline{\nu}}}{pq} + i\epsilon_{\mu\rho\lambda\delta}\,p^\lambda\,q^\delta\frac{F_3^{\nu,\overline{\nu}}}{2pq}\,. \tag{52}$$

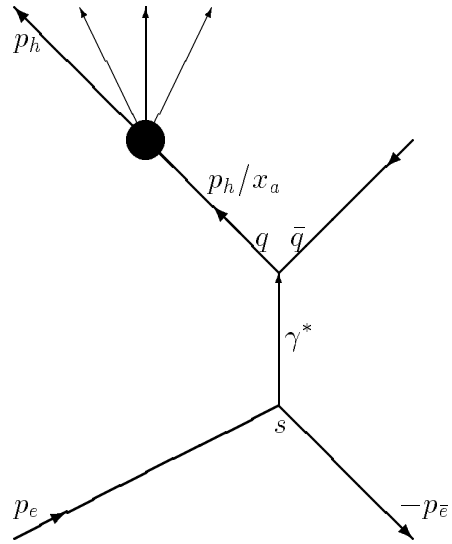

Figure 8: Inclusive $e\bar{e}$-annihilation to hadrons in the parton model

In the framework of the parton model for the neutrino scattering off proton one can obtain

$$F_1^\nu = \frac{1}{2x} F_2^\nu = d(x)\cos^2\theta_c + \overline{u}(x) + s(x)\,\sin^2\theta_c\,, \tag{53}$$

$$\frac{1}{2} F_3^\nu = d(x)\cos^2\theta_c - \overline{u}(x) + s(x)\,\sin^2\theta_c\,, \tag{54}$$

$$F_1^{\overline{\nu}} = \frac{1}{2x} F_2^{\overline{\nu}} = u(x) + \overline{d}(x)\cos^2\theta_c + \overline{s}(x)\,\sin^2\theta_c\,, \tag{55}$$

$$\frac{1}{2} F_3^{\overline{\nu}} = u(x) - \overline{d}(x)\cos^2\theta_c - \overline{s}(x)\,\sin^2\theta_c\,. \tag{56}$$

Here $u(x)$, $d(x)$, $s(x)$, $\overline{u}(x)$, $\overline{d}(x)$, $\overline{s}(x)$ are the parton distributions of the corresponding quarks in the proton and θ_c is the Cabbibo angle - the parameter of the Cabbibo-Kobayashi-Maskawa matrix (we neglect the presence of the heavier quarks in the proton). The elements of the CKM matrix appear in the vertex of the quark interaction with the W-boson.

The simplest process, in which one can measure the distribution $n_q^h(z)$ of the hadron with the relative momentum z inside the quark q is the inclusive annihilation of the e^+e^- -pair into hadrons. In this process only one hadron h with momentum p is detected. Its differential cross-section corresponding to the intermediate photon production (neglecting the Z-boson contribution) is [6] (see Fig. 8)

$$d\sigma_\gamma = \frac{4\alpha^2}{\pi}\frac{1}{s^3}\overline{L}^{\mu\nu}\overline{W}_{\mu\nu}\frac{d^3p}{E}\,,\; E = |p|\;\;,\;\; s = q^2\,, \tag{57}$$

where

$$\overline{L}^{\mu\nu} = \frac{1}{4} tr\,(\widehat{p_e}\gamma^\mu\widehat{p_{\bar{e}}}\gamma^\nu) = p_e^\mu p_{\bar{e}}^\nu + p_e^\nu p_{\bar{e}}^\mu - \delta^{\mu\nu}(p_e p_{\bar{e}}) \tag{58}$$

and

$$\overline{W}_{\mu\nu} = \frac{1}{8}\sum_n \langle 0\left|J_\mu^{el}(0)\right|n,h\rangle\,\langle n,h\left|J_\nu^{el}(0)\right|0\rangle\,(2\pi)^4\delta^4(q - p_n - p_h)\,. \tag{59}$$

We have from the gauge invariance and the parity conservation

$$\frac{1}{\pi}\overline{W}_{\mu\nu} = -\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\overline{F}_1(x_a) + \left(p_\mu - \frac{q_\mu(pq)}{q^2}\right)\left(p_\nu - \frac{q_\nu(pq)}{q^2}\right)\frac{\overline{F}_2(x_a)}{pq}\,, \tag{60}$$

where

$$x_a = \frac{2pq}{q^2} < 1\,,\; q^2 = s\,. \tag{61}$$

In the parton model the inclusive annihilation $e^+e^- \to h + ...$ is described as the process, in which initially e^+ and e^- produce the pair $q\overline{q}$ and later q or $\overline{q}$ transform into the hadron system with a measured particle h. For the structure functions $\overline{F}_1(x_a, q^2)$ and $\overline{F}_2(x_a, q^2)$ we obtain in this model:

$$\overline{F}_1(z) = -\frac{z}{2}\overline{F}_2(z) = \frac{3}{z}\sum_q e_q^2(h_q(z) + h_{\overline{q}}(z))\,, \tag{62}$$

where $h_q(z)$ and $h_{\overline{q}}(z)$ are inclusive distributions of hadrons h inside the corresponding partons q and $\overline{q}$ (they are called also the fragmentation functions). The factor 3 is related to the number of colored quarks in the fundamental representation of the gauge group $SU(3)$. Note, that the total cross-section of the e^+e^--annihilation in hadrons in the parton model behaves at large s similar to the cross-section for the e^+e^--annihilation in the $\mu^+\mu^-$ pair, which can be proved with the use of more general arguments [15].

In QCD [16] the Bjorken scaling for the structure functions is violated [17, 18, 19] and the quark and hadron distributions depend logarithmical on Q^2 in an accordance with the evolution equations [20, 22].

7.3 Evolution equations for parton distributions

In the framework of the parton model one can introduce the wave functions of the hadron in the infinite momentum frame $|\overrightarrow{p}| \to \infty$ with the following normalization condition:

$$1 = \|\Psi\|^2$$

$$\equiv \sum_n \int \prod_{i=1}^n \frac{d\beta_i d^2k_{\perp i}}{(2\pi)^2}\,|\Psi(\beta_1, k_{\perp 1}; \beta_2, k_{\perp 2}; ...; \beta_n, k_{\perp n})|^2\,\delta(1 - \sum_{i=1}^n \beta_i)\,\delta^2(\sum_{i=1}^n k_{\perp i})\,. \tag{63}$$

The wave functions $\Psi(\beta_1, k_{\perp 1}; \beta_2, k_{\perp 2}; ...; \beta_n, k_{\perp n})$ satisfy the Schrödinger equation $H\Psi = E(p)\Psi$ and contain in the perturbation theory the energy propagators simplified in the infinite momentum frame $|\vec{p}| \to \infty$ for the hadron

$$\left(E(p) - \sum_{i=1}^n E(k_i)\right)^{-1} = 2\,|\overrightarrow{p}|\left(m^2 - \sum_{i=1}^n \frac{m_i^2 + \overrightarrow{k}^2_{i\perp}}{\beta_i}\right)^{-1}. \tag{64}$$

In the renormalized field theories the integrals in the right hand side of the normalization condition for Ψ are divergent at large momenta. We regularize them by introducing the ultraviolet cut-off Λ^2 over the transverse momenta $k_{i\perp}$:

$$\left|k_{i\perp}^2\right| < \Lambda^2\,. \tag{65}$$

Note, that in gauge theories it is more convenient to use the dimensional regularization of the ultraviolet divergencies to keep the gauge invariance. In this case instead of Λ one introduces another dimensional parameter - the normalization point μ.

The amplitudes of the physical processes do not depend on Λ due to the property of the renormalizability, corresponding to a possibility to compensate the ultraviolet divergences by an appropriate choice of the bare coupling constant depending on Λ. In particular the renormalized coupling constant does not depend on Λ. In a hard process of the type of the deep-inelastic ep -scattering it is convenient to fix Λ as follows

$$\Lambda^2 \sim Q^2 \,. \tag{66}$$

In this case the strong interactions do not have any influence on the hard sub-process and we obtain the usual formulas of the parton model for the physical amplitudes. Note, that to preserve the partonic picture in the gauge theories QED and QCD one should use a physical gauge in which the virtual vector particles contain only the states with the positive norm. For the deep-inelastic ep-scattering it is convenient to chose the light-cone gauge

$$A_\mu \, q_\mu^{'} = 0 \,, \;\; q^{'} = q - \frac{q^2}{2pq} \, p \,, \;\; q'^2 = 0 \,, \tag{67}$$

where q is the virtual photon momentum and p is the proton momentum. In this gauge the propagator of the gauge boson is

$$D_{\mu\nu}(k) = -\frac{\Lambda_{\mu\nu}}{k^2} = \frac{\sum_{i=1,2} e_\mu^i(k') \, e_\nu^{i*}(k')}{k^2} + \frac{q'_\mu q'_\nu}{(kq')^2} \,, \;\; \Lambda_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu q_\nu^{'} + k_\nu q_\mu^{'}}{kq'} \,. \tag{68}$$

On the mass shell $k'^2 = 0$ ($k' = k - \frac{k^2}{xs} q'$) it contains only the physical polarization vectors $e_\mu^i(k')$ satisfying the constraints

$$e_\mu^i(k') \, k'_\mu = e_\mu^i(k') \, q'_\mu = 0 \,. \tag{69}$$

Note, that the last contribution $q'_\mu q'_\nu/(kq')^2$ to $D_{\mu\nu}(k)$ usually is not important because it does not contain the pole $1/k^2$ and sometimes the propagator is multiplied by q'_μ or q'_ν.

The polarization vectors have the following Sudakov decomposition

$$e^i = e_\perp^i - \frac{k e_\perp^i}{kq'} q' \tag{70}$$

and therefore they are parametrized through their transverse components.

The Λ-dependence of the partonic wave functions expressed in terms of the physical charge is determined by the renormalization group:

$$|\Psi(\beta_1, k_{\perp 1}; \beta_2, k_{\perp 2}; ...; \beta_n, k_{\perp n}|^2 \sim \prod_r Z_r^{n_r} \,. \tag{71}$$

Here $\sqrt{Z_r} < 1$ are the renormalization constants for the wave functions of the corresponding fields and Z_r is the probability to find the physical particle r in the corresponding one-particle bare state. Further, n_r is the number of the bare particles r in the state of n partons. For finding the Λ-dependence of the limits of integration in $k_{\perp i}$ one should take into account, that the largest contribution in the normalization condition occurs from the momentum configuration of the type of the Russian doll, when the constituent particles in

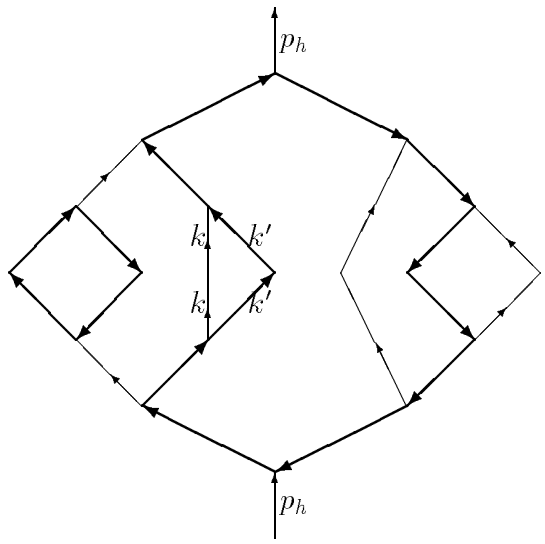

Figure 9: Normalization condition for the partonic wave function

the initial hadron consist of two partons, each of these partons again consists of two other partons and so on. In each step of this parton branching the transverse momenta of the particles grow and only one of the last partons in this chain of decays reaches the largest possible value $|k_\perp| = \Lambda$. It is related to the fact, that only for such configuration the number of the energy propagators with large denominators is minimal. Moreover, the quantum-mechanical interference of amplitudes with different schemes of decays is not important in the normalization condition within the leading logarithmic accuracy in $\ln \Lambda$ (see Fig. 9).

Therefore if we shall differentiate the normalization condition in Λ, the most essential contribution will appear from the upper limits of the integrals over $k_\perp^2 \simeq k_\perp'^2$ for two partons p, p' produced in the end of the decay chain (see Fig. 9) and one can obtain from this differentiation the equation

$$0 = \sum_r \overline{n_r} \left(\frac{d \ln Z_r}{d \ln(\Lambda^2)} + \gamma_r \right) \, , \; \gamma_r = \sum_{p,p'} \frac{d \left\| \Psi_{r \to pp'} \right\|^2}{d \ln(\Lambda)^2} \, . \tag{72}$$

Here $\left\| \Psi_{r \to pp'} \right\|^2 \sim g^2$ is the one-loop contribution to the norm of the wave function of the parton r related to its transition to the two particles p and p'. The anomalous dimension γ_r depends on the coupling constant, which correspons to its different normalization in comparison with the previous Chapter. The quantity

$$\overline{n_r} = \sum_{\{n_s\}} n_r \int \prod_{i=1}^{n} \frac{d\beta_i d^2 k_{\perp i}}{(2\pi)^2} \left| \Psi(\beta_1, k_{\perp 1}; \beta_2, k_{\perp 2}; ...; \beta_n, k_{\perp n} \right|^2 \delta(1 - \sum_{i=1}^{n} \beta_i) \, \delta^2(\sum_{i=1}^{n} k_{\perp i}) \tag{73}$$

is an averaged number of partons r in the hadron. Because this number is different in different hadrons, we should have

$$\frac{dZ_r}{d \ln(\Lambda^2)} = -\gamma_r Z_r \, , \tag{74}$$

which coincides with the Callan-Simanzik equation for the renormalization constants. Because $\gamma_r > 0$, for $\Lambda^2 \to \infty$ we have $Z_r \to 0$, which means, that the probability to find a physical particle in its bare state is zero.

In an analogous way, after the differentiation of the partonic expression for the density of the number of partons k in the hadron

$$n_r(x) = \sum_{\{n_s\}} \int \prod_{i=1}^{n} \frac{d\beta_i d^2 k_{\perp i}}{(2\pi)^2} \, |\Psi(\beta_1, k_{\perp 1}; ...; \beta_n, k_{\perp n}|^2 \, \delta(1 - \sum_{i=1}^{n} \beta_i) \, \delta^2(\sum_{i=1}^{n} k_{\perp i}) \sum_{i \in r} \delta(\beta_i - x) \, , \quad (75)$$

with the use of the Callan-Simanzik equation we shall obtain for $n_r(x)$ the evolution equations of Dokshitzer, Gribov, Lipatov, Altarelli and Parisi (DGLAP) [20].

In QCD the DGLAP equations have the form

$$\frac{d}{d\xi} n_k(x) = -w_k \, n_k(x) + \sum_r \int_x^1 \frac{dy}{y} \, w_{r \to k} \left(\frac{x}{y} \right) \, n_r(y) \; , \quad (76)$$

where w_r is proportional to γ_r:

$$w_r = \sum_k \int_0^1 dx \, x \, w_{r \to k}(x) \, . \quad (77)$$

The last result can be obtained from the energy conservation

$$\sum_k \int_0^1 d \, x \, x \, n_k(x) = 1 \, . \quad (78)$$

The variable ξ

$$\xi = -\frac{2 N_c}{\beta_s} \ln \frac{\alpha(Q^2)}{\alpha_\mu} \quad (79)$$

is related to the QCD running coupling constant [17]

$$d\xi = \frac{\alpha(Q^2) \, N_c}{2\pi} \, d \ln Q^2 \, , \;\; \alpha(Q^2) = \frac{\alpha_\mu}{1 + \beta_s \frac{\alpha_\mu}{4\pi} \ln \frac{Q^2}{\mu^2}} \, , \; \beta_s = \frac{11}{3} N_c - n_f \frac{2}{3} \; . \quad (80)$$

The DGLAP equation has a simple probabalistic interpretation and it is analogous to the balance equation for the densities of various gases being in a chemical equilibrium. Indeed, the first term in its right hand side describes decreasing the number of partons k as a result of their decay to other partons in the opening phase space $d\xi$. On the other hand, the second term is responsible for its decreasing due to the fact, that other partons r can decay in the states containing the particles k.

By integrating over x one can obtain simpler evolution equations for an averaged number of partons:

$$\frac{d}{d\xi} n_k = -w_k \, n_k + \sum_r w_{r \to k} \; n_r \; , \;\; w_{r \to k} = \int_0^1 dx \, w_{r \to k} \, (x) \; . \quad (81)$$

for the average number of partons

$$n_k = \int_0^1 d \, x \, n_k(x) \, . \quad (82)$$

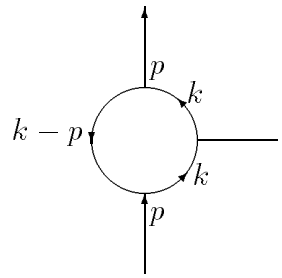

Figure 10: Matrix element of the conserved current or energy-momentum tensor

For the electric charge and other additive quantum numbers we have the conservation law

$$\frac{d}{d\xi}\sum_k Q_k\, n_k = 0 \tag{83}$$

due to the following property

$$\sum_k Q_k\, w_{r\to k} = w_r\, Q_r \tag{84}$$

of the transition probabilities. The energy conservation

$$\frac{d}{d\xi}\sum_k \int_0^1 dx\, x\, n_k(x) = 0 \tag{85}$$

is valid due to the following property of the splitting kernels

$$\sum_k \int_0^1 dx\, x\, w_{r\to k}(x) = w_r\,. \tag{86}$$

In turn, these sum rules can be applied for finding the splitting kernels $w_{r\to i}(x)$. For this purpose on one hand we should use the Feynman diagram approach to calculate the matrix elements of the conserved currents $j_\mu(z)$ or energy-momentum stress tensor $T_{\mu\nu}(z)$ in one-loop approximation (see Fig. 10). On the other hand, with the use of the Sudakov parametrization

$$k = \alpha q' + xp + k_\perp \tag{87}$$

of momenta of the virtual particles we can write for the matrix elements of the current $j_\mu\, q'_\mu$

$$\xi s' \int_0^1 dx\;\, w_{r\to i}(x) = i\int \frac{|s'|\, d\alpha\, dx\, d^2k_\perp}{2\,(2\pi)^4}\, \frac{g^2(k_\perp^2)\,\sum_t\, |\gamma_{r\to i,t}|^2}{-s(1-x)\alpha - \vec{k}_\perp^{\,2} + i\varepsilon}\, \frac{s'x}{\left(sx\alpha - \vec{k}_\perp^{\,2} + i\varepsilon\right)^2} \tag{88}$$

in the case, when the particles r and i have the same conserved quantum number Q, and

$$\xi\, s'^2 \int_0^1 dx\, x\, w_{r\to i}(x) = i\int \frac{|s'|\, d\alpha\, dx\, d^2k_\perp}{2(2\pi)^4}\, \frac{g^2(k_\perp^2)\sum_t\, |\gamma_{r\to i,t}|^2}{-s(1-x)\alpha - \vec{k}_\perp^{\,2} + i\varepsilon}\, \frac{s'^2x^2}{\left(sx\alpha - \vec{k}_\perp^{\,2} + i\varepsilon\right)^2}\,. \tag{89}$$

for the energy-momentum tensor $T_{\mu\nu}q'_\mu q'_\nu$. Here $\gamma_{r\to i,t}$ is the corresponding transition amplitude calculated in the helicity basis. This basis is convenient, because the helicity is

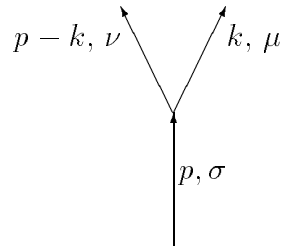

Figure 11: Yang-Mills vertex

conserved for the matrix elements of j_μ and $T_{\mu\nu}$. We find below the amplitudes $\gamma_{r\to i,t}$ for all possible parton transitions in QCD. In the above expressions the integrals over α are non-zero only if the Sudakov variable x belongs to the interval $0 < x < 1$. They can be calculated by residues leading to the following expressions for splitting kernels

$$w_{r\to i}(x) = \sum_t \frac{\left|\gamma_{r\to i,t}\right|^2}{N_c} \frac{x(1-x)}{2\left|\overrightarrow{k}_\perp\right|^2}\,, \tag{90}$$

which do not depend on $\overrightarrow{k}_\perp$.

7.4 Splitting kernels in the Born approximation

7.4.1 Transition from a gluon to two gluons

We start with the discussion of the splitting kernels $w_{r\to k}\left(\frac{x}{y}\right)$ for the transition from gluon to gluon. The gluon Yang-Mills vertex for the transition $(p,\sigma)\to(k,\mu)+(p-k,\nu)$ can be written as follows (see Fig. 11)

$$\gamma_{\sigma\mu\nu} = (p-2k)_\sigma\,\delta_{\mu\nu} + (p+k)_\nu\,\delta_{\sigma\mu} + (k-2p)_\mu\,\delta_{\nu\sigma} \tag{91}$$

up to the colour factor f_{abc} being the structure constant of the color group $SU(N_c)$ entering in the commutation relations of the generators:

$$[T_a, T_b] = i f_{abc}\, T_c\,. \tag{92}$$

On the mass shell the numerators of the gluon propagators in the light-cone gauge coincide with the projectors to physical states. The corresponding polarization vectors satisfying also the Lorentz condition $ek' = 0$ have the following Sudakov representation

$$e_\sigma(p) = e^\perp_\sigma\,,\ e_\mu(k') = e^\perp_\mu - \frac{ke^\perp}{kq'}\,q'_\mu\,,\ e_\nu(p-k') = e^\perp_\nu - \frac{(p-k)e^\perp}{(p-k)q'}\,q'_\nu\,,\ k' = k - \frac{k^2}{2kq'}q'\,. \tag{93}$$

After the multiplication of the Yang-Mills vertex $\gamma_{\sigma\mu\nu}$ with these polarization vectors one can introduce the tensor with transverse components according to the definition

$$\gamma_{\sigma\mu\nu}\,e^\sigma(p)e^\mu(k')e^\nu(p-k') = \gamma^\perp_{\sigma\mu\nu}\,e^\sigma_\perp(p)e^\mu_\perp(k')e^\nu_\perp(p-k')\,, \tag{94}$$

where

$$\gamma^{\perp}_{\sigma\mu\nu} = -2k^{\perp}_{\sigma}\,\delta^{\perp}_{\mu\nu} + \frac{2}{1-x}\,k^{\perp}_{\nu}\,\delta^{\perp}_{\sigma\mu} + \frac{2}{x}\,k^{\perp}_{\mu}\,\delta^{\perp}_{\nu\sigma}\,,\;\; x = \frac{kp}{q'p}\,. \tag{95}$$

The gluons moving in the z-direction with the helicities $\lambda = \pm$ are described by the polarization vectors

$$e^{\pm}_{\perp} = \frac{1}{\sqrt{2}}(e^1 \pm ie^2)\,. \tag{96}$$

We put the helicity λ of the initial gluon with the momentum p equal $+1$, because for $\lambda = -1$ the results can be found from the obtained expressions by changing the signs of helicities of the final particles. Then the non-zero matrix elements

$$\gamma_{\lambda_1\lambda_2} = \gamma^{\perp}_{\sigma\mu\nu}\; e^{\sigma}_{+}(p) e^{*\mu}_{\lambda_1}(k') e^{*\nu}_{\lambda_2}(p-k') \tag{97}$$

are

$$\gamma_{++} = \sqrt{2}\frac{k^*}{x(1-x)}\,,\; \gamma_{+-} = \sqrt{2}\frac{x}{1-x}\,k\,,\;\; \gamma_{-+} = \sqrt{2}\frac{1-x}{x}\;k\,, \tag{98}$$

where $k = k_1 + ik_2$, $k^* = k_1 - ik_2$.

The dimensionless quantities

$$w_{1^+\to i}(x) = \frac{x(1-x)}{2\,|k|^2}\sum_{t=\pm}|\gamma_{it}|^2\;, \tag{99}$$

are the splitting kernels (we do not write the color factor $f_{abc}f_{a'bc} = N_c\delta_{aa'}$, because it is included in the definition of ξ). They equal

$$w_{1^+\to 1^+}(x) = \frac{1+x^4}{x(1-x)}\,,\;\; w_{1^+\to 1^-}(x) = \frac{(1-x)^4}{x(1-x)}\,. \tag{100}$$

In the contribution to $w_g = w_{g\to gg} + w_{g\to q\bar{q}}$ one can obtain from the transition to gluons

$$w_{g\to gg} = \frac{1}{2}\int_0^1 \frac{1+x^4+(1-x)^4}{x(1-x)}\;dx\;, \tag{101}$$

where we substituted the factor x by $\frac{1}{2}$ in the integrand due to its symmetry to the substitution $x \to 1-x$. The divergency of $w_{g\to gg}$ at $x = 0, 1$ is cancelled in the evolution equations. There is also the contribution to w_g from the quark-anti-quark state (see below):

$$w_{g\to q\bar{q}} = \frac{n_f}{2N_c}\int_0^1 \left(x^2 + (1-x)^2\right)\;dx\; = \frac{n_f}{3N_c}\,. \tag{102}$$

The matrix elements for the anomalous dimension matrix describing gluon-gluon transitions can be written as follows

$$w^j_{r\to k} = \int_0^1 dx\;\; w_{r\to k}(x)\;\left(x^{j-1} - x\,\delta_{rk}\right) - \frac{n_f}{3N_c}\;\delta_{rk}\,. \tag{103}$$

Thus, we obtain

$$w^j_{1^+\to 1^+} = 2\psi(1) - 2\psi(j-1) - \frac{1}{j+2} - \frac{1}{j+1} - \frac{1}{j} - \frac{1}{j-1} + \frac{11}{6} - \frac{n_f}{3N_c}\quad . \tag{104}$$

and

$$w^{j}_{1^+\to 1^-} = -\frac{1}{j+2} + \frac{3}{j+1} - \frac{3}{j} + \frac{1}{j-1}\,. \tag{105}$$

The anomalous dimension of tensors $G_{\mu_1\sigma}D_{\mu_2}...G_{\mu_j\sigma}$

$$w^{jv}_{1\to 1} = 2\psi(1) - 2\psi(j-1) - \frac{2}{j+2} + \frac{2}{j+1} - \frac{4}{j} + \frac{11}{6} - \frac{n_f}{3N_c}\,. \tag{106}$$

Note, that the regular term $\frac{11}{6} - \frac{n_f}{3N_c}$ is proportional to the function β_s entering in the expression for the running of the QCD coupling constant. It seems to be related to a supersymmetric generalization of QCD.

Further, the anomalous dimension for the axial tensors $G_{\mu_1\sigma}D_{\mu_2}...\widetilde{G}_{\mu_j\sigma}$ in the gluodynamics is

$$w^{ja}_{1\to 1} = 2\psi(1) - 2\psi(j) - \frac{4}{j+1} + \frac{2}{j} + \frac{11}{6} - \frac{n_f}{3N_c}. \tag{107}$$

The energy conservation sum rule for $j = 2$

$$w^{2v}_{1\to 1} + w^{2v}_{1\to 1/2} = 0. \tag{108}$$

is fulfilled as it can be verified from the expression for $w^{jv}_{1\to 1/2}$ obtained in the next section. Because the contribution $w^{jv}_{1\to 1/2}$ is proportional to n_f, one can verify that $w^{2v}_{1\to 1} = 0$ at $n_f = 0$.

7.4.2 Transition from a gluon to a quark pair

The propagator of the massless fermion can be written in the form

$$G(k) = \frac{\widehat{k}}{k^2}\,,\ \ \widehat{k} = \sum_{\lambda=\pm} u^{\lambda}(k')\,\overline{u^{\lambda}(k')} + \frac{k^2}{2kq'}\,\widehat{q}'\,,\ \ k' = k - \frac{k^2}{xs}q'\,, \tag{109}$$

where $k'^2 = 0$. The last contribution in $\widehat{k}$ is absent on the mass shell or providing that the vertex neighboring to the propagator G is $\widehat{q}'$. The massless fermion with the momentum $\overrightarrow{k}$ and the helicity $\lambda/2$ is described by the spinor

$$u^{\lambda}(\overrightarrow{k}) = \sqrt{k_0}\begin{pmatrix} \varphi^{\lambda} \\ \lambda\varphi^{\lambda} \end{pmatrix}\,,\ \ \overline{u}\,\gamma_{\mu}\,u = 2\,k_{\mu}\,,\ \widehat{k} = \sum_{\lambda} u^{\lambda}\overline{u}^{\lambda}\,, \tag{110}$$

where the Pauli spinor φ satisfies the equation

$$\overrightarrow{\sigma}\,\overrightarrow{k}\,\varphi^{\lambda} - \lambda\,k_0\,\varphi^{\lambda} = \begin{pmatrix} k_3 - \lambda\,k_0 & k_1 - ik_2 \\ k_1 + ik_2 & -k_3 - \lambda\,k_0 \end{pmatrix}\begin{pmatrix} \varphi^{\lambda}_1 \\ \varphi^{\lambda}_2 \end{pmatrix} = 0\,,\ \ k_0 = \left|\overrightarrow{k}\right|. \tag{111}$$

In the light-cone frame $p = p_3 \to \infty$ we have

$$\varphi^{+} \simeq \begin{pmatrix} 1 \\ \frac{k}{2\,x\,p} \end{pmatrix}\,,\ \varphi^{-} \simeq \begin{pmatrix} \frac{-k^*}{2\,x\,p} \\ 1 \end{pmatrix}\,. \tag{112}$$

The massless anti-fermion with the momentum $\overrightarrow{p} - \overrightarrow{k}$ and the helicity $\lambda'/2$ is described by the spinor

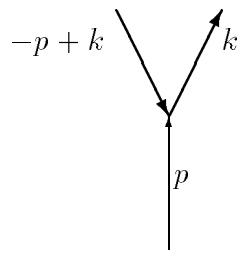

Figure 12: Gluon-quark-anti-quark vertex

$$v^{\lambda'}(-\overrightarrow{p}+\overrightarrow{k}) = \sqrt{p_0-k_0}\begin{pmatrix} \chi^{\lambda'} \\ -\lambda'\chi^{\lambda'} \end{pmatrix}, \tag{113}$$

where the Pauli spinor χ satisfies the equation

$$\overrightarrow{\sigma}(\overrightarrow{p}-\overrightarrow{k})\chi^{\lambda'} + \lambda'(p_0-k_0)\,\chi^{\lambda'} = 0\,. \tag{114}$$

In the light-cone frame we have

$$\chi^- \simeq \begin{pmatrix} 1 \\ \frac{-k}{2(1-x)\,p} \end{pmatrix}, \ \chi^+ \simeq \begin{pmatrix} \frac{k^*}{2\,(1-x)\,p} \\ 1 \end{pmatrix} \tag{115}$$

and v^λ satisfies the equation $\gamma_5 v^\lambda = -\lambda v^\lambda$. Note, that in the case of the left Pauli particle - neutrino we have only $\lambda = -1$ for ν and $\lambda' = 1$ for $\overline{\nu}$ in a correspondence with the eigenvalues of the matrix γ_5 for its eigenfunctions u^λ and $v^{\lambda'}$, respectively.

Thus, for the matrix element of the vertex describing the transition of the gluon with its momentum $\overrightarrow{p}$ and helicity 1 into a pair of fermions (see Fig. 12) we have

$$\gamma^{\lambda\lambda'} = \overline{u}^\lambda(\overrightarrow{k})\,\frac{\gamma^1+i\gamma^2}{\sqrt{2}}\,v^{\lambda'}(-\overrightarrow{p}+\overrightarrow{k}) = 2\sqrt{(p_0-k_0)\,k_0}\lambda\,\delta_{\lambda,-\lambda'}\ \sqrt{2}\varphi^{\lambda *}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\chi^{\lambda'}. \tag{116}$$

The non-zero matrix elements $\gamma^{\lambda\lambda'}$are given below

$$\gamma^{+-} = -\sqrt{2}\,k\sqrt{\frac{x}{1-x}}\,,\ \ \gamma^{-+} = -\sqrt{2}\,k\sqrt{\frac{1-x}{x}}\,. \tag{117}$$

Now let us use the following representation for splitting kernels, which was obtained above,

$$w_{1^+\to q}(x) = \frac{n_f}{2N_c}\frac{x(1-x)}{2\,|k|^2}\sum_t \left|\gamma^{qt}\right|^2\,. \tag{118}$$

Here $\frac{1}{2}$ is the color factor, n_f is the number of different types of quarks and the factor $1/N_c$ takes into account, that N_c is included in the definition of ξ. Thus, we obtain

$$w_{1^+\to q^+}(x) = w_{1^+\to\overline{q}^+}(x) = \frac{n_f}{2N_c}x^2\,,\ w_{1^+\to q^-}(x) = w_{1^+\to\overline{q}^-}(x) = \frac{n_f}{2N_c}(1-x)^2\,. \tag{119}$$

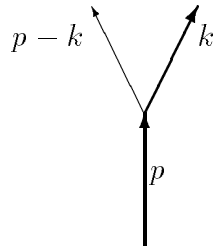

Figure 13: Quark-gluon-quark vertex

The total probability for the gluon transition to quark is

$$w_{1\to q,\overline{q}} = \frac{n_f}{2N_c}\int_0^1 \left(x^2+(1-x)^2\right)\ dx\ = \frac{n_f}{3N_c}\,, \tag{120}$$

For QED the color factor $\frac{n_f}{2N_c}$ is absent and we obtain $w_{1\to\frac{1}{2}\frac{1}{2}} = \frac{2}{3}$. Taking into account, that in the perturbation theory $\xi = \frac{\alpha}{2\pi}\ln\Lambda^2$ for QED, we obtain, that the probability for the photon to be in the e^+e^- state is $\frac{\alpha}{3\pi}\ln\Lambda^2$ in an agreement with the known result for the charge renormalization in this theory.

The non-vanishing matrix elements for the anomalous dimension matrix are

$$w^j_{1^+\to q^+} = \frac{n_f}{2N_c}\frac{1}{j+2}\,,\ w^j_{1^+\to q^-}(x) = \frac{n_f}{2N_c}\left(\frac{1}{j}-\frac{2}{j+1}+\frac{1}{j+2}\right) \tag{121}$$

and these elements for the vector and axial current are

$$w^{jv}_{1\to 1/2} = \frac{n_f}{N_c}\left(\frac{1}{j}-\frac{2}{j+1}+\frac{2}{j+2}\right)\,, \tag{122}$$

$$w^{ja}_{1\to 1/2}(x) = \frac{n_f}{N_c}\left(-\frac{1}{j}+\frac{2}{j+1}\right)\,, \tag{123}$$

where we added the gluon transitions to the quark and anti-quark.

7.4.3 Transition from a quark to the quark–gluon system

The amplitude for the transition of a quark with the helicity + to quark and gluon (see Fig. 13) can be written as follows in the light cone gauge for the gluon polarization vector:

$$\overline{u}^+(k)\,(\widehat{e^*}^\lambda_\perp + \frac{k_\perp e^{*\lambda}}{(p-k,q')}\,\widehat{q}')\,u^+(p) = \sqrt{k_0p_0}2\varphi^{+*}\overrightarrow{\sigma}\overrightarrow{e^{*\lambda}}\varphi^+ + 2\,k_\perp e^{*\lambda}\,\frac{\sqrt{x}}{1-x}\,. \tag{124}$$

Therefore we have

$$\overline{u}^+(k)\,(\widehat{e}^{+*}_\perp + \frac{k_\perp e^{+*}}{(p-k,q')}\,\widehat{q}')\,u^+(p) = \sqrt{2}k^*\frac{1}{\sqrt{x}(1-x)}\,, \tag{125}$$

$$\overline{u}^+(k)\,(\widehat{e}^{-*}_\perp + \frac{k_\perp e^{-*}}{(p-k,q')}\,\widehat{q}')\,u^+(p) = \sqrt{2}k\frac{\sqrt{x}}{(1-x)}\,. \tag{126}$$

Thus, the splitting kernels are

$$w_{1/2^+\to 1^+}(x) = \frac{c}{x}\,,\ w_{1/2^+\to 1^-}(x) = c\frac{(1-x)^2}{x}\,,\ w_{1/2^+\to 1/2^+}(x) = c\frac{1+x^2}{1-x}\,, \tag{127}$$

where

$$c = \frac{N_c^2-1}{2N_c^2} \tag{128}$$

is the color factor for the corresponding loop (note, that N_c is included in ξ)

The total contribution to $w_{1/2}$ is

$$w_{1/2} = \frac{c}{2}\int_0^1 \left(\frac{1+x^2}{1-x} + \frac{1+(1-x)^2}{x}\right)\ dx\ . \tag{129}$$

The corresponding anomalous dimensions are

$$w^j_{1/2^+\to 1^+} = \frac{c}{j-1}\,,\ w^j_{1/2^+\to 1^-} = c\left(\frac{1}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right)\,, \tag{130}$$

$$w^j_{1/2^+\to 1/2^+} = c\int_0^1 \frac{1+x^2}{1-x}\,(x^{j-1}-1)\ dx = c\left(2\psi(1) - 2\psi(j) - \frac{1}{j+1} - \frac{1}{j} + \frac{3}{2}\right)\,. \tag{131}$$

The vector and axial contributions are

$$w^{jv}_{1/2\to 1} = c\left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right)\,, \tag{132}$$

$$w^{ja}_{1/2\to 1} = c\left(\frac{2}{j} - \frac{1}{j+1}\right)\,. \tag{133}$$

We have the sum rules

$$w^{1v}_{1/2\to 1/2} = 0\,,\ w^{2v}_{1/2\to 1} + w^{2v}_{1/2\to 1/2} = 0, \tag{134}$$

expressing the conservation of the barion charge and the energy, respectively, in the quark decay.

As it is seen from the above formulae, there are two relations among the matrix elements of the anomalous dimension matrix for $n_f = N_c$

$$\frac{1}{c}\left(w^{jv}_{1/2\to 1} + w^{jv}_{1/2\to 1/2}\right) = w^{jv}_{1\to 1} + \ w^{jv}_{1\to 1/2} = 2\psi(1) - 2\psi(j-1) - \frac{3}{j} + \frac{3}{2}\,, \tag{135}$$

$$\frac{1}{c}\left(w^{ja}_{1/2\to 1} + w^{ja}_{1/2\to 1/2}\right) = w^{ja}_{1\to 1} + \ w^{ja}_{1\to 1/2} = 2\psi(1) - 2\psi(j) - \frac{2}{j+1} + \frac{1}{j} + \frac{3}{2}\,, \tag{136}$$

These relations can be derived using arguments based on the super-symmetry (SUSY). In the super-symmetric generalization of the Yang-Mills theory the gluon and its partner - gluino are unified in one multiplet. The gluino is a Maiorano fermion which coincides with the corresponding anti-particle and belongs to an adjoint representation of the gauge group. In this model the total probability to find both gluon and gluino with the parameter

x should not depend on the spin $s = 1, 1/2$ of the initial particle being a component of the super-multiplet.

On the other hand, for the probability of the gluon transition to the $q\overline{q}$ pair in QCD we have the extra factor n_f/N_c in comparison with its transition to two gluinos, because the ratio of the number of the fermion states for these theories is $2n_f$ and the ratio of the corresponding color factors is $1/(2N_c)$. Analogously, for the probability of the transition of the quark to the gluon and quark we have the additional color factor $c = \frac{N_c^2-1}{2N_c}/N_c$–the ratio of two Kazimir operators in comparison with the transition to the gluon and gluino.

7.5 Evolution equations for fragmentation functions

Let us consider now evolution equations for the fragmentation functions $\bar{D}_i^h(z) = h_i(z)$ which are the distributions of the hadron h with the momentum being the longitudinal fraction z of the momentum of the parton i. The quark fragmentation function $h_q(z)$ enters for example in the cross-section for the inclusive production of hadrons (see (57)). The initial parton is assumed to be a highly virtual particle. The virtualities of the particles decrease in the process of their decay to other partons (cf. Fig.(9)) in such way, that close to the moment of the hadron production their virtualities are of the order of value of the characteristic QCD scale Λ_{QCD}.

To find the normalization condition for the wave function $\psi_i(k_1, .., k_n)$ of the virtual parton i with momentum k being a superposition of the near-mass shell partons with momenta $k_1, .., k_n$ let us consider the non-renormalized Green functions in the light-cone gauge. One can present such Green functions for the quark and gluon in the form (cf. (68) and (109))

$$G(k) = \frac{\hat{k}}{k^2} g(k^2, kq') + \frac{\hat{q}'}{kq'} f(k^2, kq') \,,$$

$$D_{\mu\nu}(k) = \frac{\Lambda_{\mu\nu}}{k^2}\, d(k^2, kq') + \frac{q'_\mu\, q'_\nu}{(kq')^2}\, c(k^2, kq') \,, \tag{137}$$

where we used the light-cone vector q' (67). In our kinematics we have for the parton virtuality (see (65) and (66)) the restriction from above

$$k^2 \ll 2kq' \sim \Lambda^2 \sim Q^2 \,, \tag{138}$$

where Q^2 is a characteristic large scale in the corresponding hard process. For example, in the e^+e^- annihilation to hadrons this scale is the virtuality of the intermediate photon. The functions g, f, d and c depend in the leading logarithmic approximation only on the ratio k^2/Λ^2.

Because at $k^2 = \Lambda^2$ the non-renormalized Green functions coincide with the free propagators, we have the following normalization conditions

$$g(k^2/\Lambda^2)_{|_{k^2=\Lambda^2}} = 1 \,,\ f(k^2/\Lambda^2)_{|_{k^2=\Lambda^2}} = 0 \,,$$

$$d(k^2/\Lambda^2)_{|_{k^2=\Lambda^2}} = 1 \,,\ c(k^2/\Lambda^2)_{|_{k^2=\Lambda^2}} = 0 \,. \tag{139}$$

Let us write the dispersion relations for the functions g and d in the form

$$\frac{1}{k^2}\, g(k^2/\Lambda^2) = \frac{z_q}{k^2} + \frac{1}{\pi} \int_{m^2}^{\Lambda^2} \frac{d\,\tilde{k}^2}{k^2 - \tilde{k}^2 + i\epsilon}\, \Im\left(-\frac{1}{\tilde{k}^2}\, g(\tilde{k}^2/\Lambda^2)\right) \,,$$

$$\frac{1}{k^2}\, d(k^2/\Lambda^2) = \frac{z_g}{k^2} + \frac{1}{\pi}\int_{m^2}^{\Lambda^2} \frac{d\,\tilde{k}^2}{k^2 - \tilde{k}^2 + i\epsilon}\, \Im\left(-\frac{1}{\tilde{k}^2}\, d(\tilde{k}^2/\Lambda^2)\right)\,, \tag{140}$$

where z_q and z_g are squares of the renormalization constants for corresponding fields.

From above constraints (139) we obtain the following sum rules

$$1 = z_q + \frac{1}{\pi}\int_{m^2}^{\Lambda^2} d\,\tilde{k}^2\, \Im\left(-\frac{1}{\tilde{k}^2}\, g(\tilde{k}^2/\Lambda^2)\right)\,,$$

$$1 = z_g + \frac{1}{\pi}\int_{m^2}^{\Lambda^2} d\,\tilde{k}^2\, \Im\left(-\frac{1}{\tilde{k}^2}\, d(\tilde{k}^2/\Lambda^2)\right)\,, \tag{141}$$

They correspond to the normalization conditions for the wave functions $\psi_i(k_1,..,k_n)$ which will be derived below.

For this purpose let us find g and d from eqs. (137)

$$g(k^2/\Lambda^2) = \frac{k^2}{2kq'}\,\frac{1}{2}\, Sp\, \hat{q}' G(k)\,,\ d(k^2/\Lambda^2) = \frac{1}{2}\,\Lambda_{\mu\nu}\, D_{\mu\nu}(k)\,. \tag{142}$$

In these relations one can express the propagators through the projectors to the physical states (see (68) and (109))

$$D_{\mu\nu}(k) \sim \frac{\sum_{i=1,2} e^i_\mu(k')\, e^{i*}_\nu(k')}{k^2}\,,\ G(k) \sim \frac{\sum_{\lambda=\pm} u^\lambda(k')\,\overline{u^\lambda(k')}}{k^2}\,,\ k' = k - \frac{k^2}{xs}\, q'\,, \tag{143}$$

because the non-physical contributions are cancelled. Note, that these representations are in an agreement with the normalization conditions for Dirac spinors and polarization vectors

$$\frac{1}{2kq'}\,\overline{u^\lambda(k')}\,\hat{q}'\, u^{\lambda'}(k') = \delta^{\lambda\lambda'}\,,\ e^\lambda_\mu(k')\, e^{\lambda'*}_\nu(k')\,\Lambda_{\mu\nu} = \delta^{\lambda\lambda'}\,. \tag{144}$$

In the relations (140) and (141) one can use the following expressions for the imaginary parts of the corresponding Green functions in terms of the amplitudes $M^\lambda_{i\to h_n}$ describing transitions of the initial parton i to the hadron states h_n

$$2\Im\left(-\frac{1}{\tilde{k}^2}\, g(\tilde{k}^2/\Lambda^2)\right) = \frac{1}{\tilde{k}^4}\,\frac{1}{2}\sum_{\lambda=\pm 1/2}\sum_{h_n}\int d\Omega_n |M^\lambda_{q\to h_n}|^2\,,$$

$$2\Im\left(-\frac{1}{\tilde{k}^2}\, d(\tilde{k}^2/\Lambda^2)\right) = \frac{1}{\tilde{k}^4}\,\frac{1}{2}\sum_{\lambda=\pm 1}\sum_{h_n}\int d\Omega_n |M^\lambda_{g\to h_n}|^2\,, \tag{145}$$

where we averaged over the helicities $\pm s$ of the initial parton. Note, that one can omit this averaging due to the parity conservation in strong interactions. The phase space $d\Omega_n$ for momenta of the final hadrons in the Sudakov variables (10) can be written as follows

$$d\Omega_n = \prod_{i=1}^{n}\frac{d^3\mathbf{k}_i}{2|\mathbf{k}_i|\,(2\pi)^3}(2\pi)^4\,\delta^4(\tilde{k} - \sum_{i=1}^{n} k_i)$$

$$= \prod_{i=1}^{n-1}\frac{d\beta_i\, d^2k_{i\perp}}{2\beta_i\,(2\pi)^3}\,\frac{2\pi}{1-\sum_{i=1}^{n-1}\beta_i}\,\delta(\tilde{k}^2 - \sum_i \frac{k^2_{i\perp}}{\beta_i})\,, \tag{146}$$

where

$$1 > \beta_i > 0\,,\ \tilde{\beta} = 1 - \sum_{i=1}^{n}\beta_i\,,\ \tilde{\alpha}s = \tilde{k}^2 < \Lambda^2\,. \tag{147}$$

From eqs. (141) and (145) one can obtain the following normalization condition for the wave function of the parton in the space of hadron states

$$1 - z_q = \frac{1}{2} \sum_{\lambda=\pm 1/2} \sum_{h_n} \int \prod_{i=1}^{n-1} \frac{d\beta_i \, d^2 k_{i\perp}}{2\beta_i \, (2\pi)^3} \frac{2\pi}{1 - \sum_{i=1}^{n-1} \beta_i} \frac{|M^{\lambda}_{q\to h_n}|^2}{\left(\sum_i \frac{k^2_{i\perp}}{\beta_i}\right)^2} ,$$

$$1 - z_g = \frac{1}{2} \sum_{\lambda=\pm 1/2} \sum_{h_n} \int \prod_{i=1}^{n-1} \frac{d\beta_i \, d^2 k_{i\perp}}{2\beta_i \, (2\pi)^3} \frac{2\pi}{1 - \sum_{i=1}^{n-1} \beta_i} \frac{|M^{\lambda}_{g\to h_n}|^2}{\left(\sum_i \frac{k^2_{i\perp}}{\beta_i}\right)^2} . \tag{148}$$

Note, that above expressions are applied strictly speaking only to the renormalized field theories of the type of QED, for which the bare particles (partons) coincide in their quantum numbers with physical particles (hadrons). In QCD quarks and gluons do not exist in the free state and therefore the amplitude $M^{\lambda}_{q\to h_n}$ does not have any physical meaning. Nevertheless, we shall use the above expressions also in the QCD case assuming, that the final hadron states include necessarily also soft quarks and gluons, which are annihilated with other soft particles appearing from the decay of the other hard partons produced in the initial state.

The kinematics in which one can obtain the leading logarithmic terms for the fragmentation functions is different from that giving the main contribution to the normalization of the partonic wave function of a hadron (see Fig.(9)). We shall discuss it below.

In an one-loop approximation ($n = 2$) for the transitions $i \to (j, k)$ the normalization condition for the parton wave function takes the form

$$1 - z_i = \sum_{\lambda_j, \lambda_k} \int_0^1 \frac{d\beta}{\beta(1-\beta)} \int \frac{d^2 k_\perp}{\left(\frac{k^2_\perp}{\beta} + \frac{k^2_\perp}{1-\beta}\right)^2} |M^{\lambda_i;\lambda_j,\lambda_k}_{i\to j,k}|^2 , \tag{149}$$

where $k, k' - k$ are momenta of two final partons and λ_j, λ_k are their polarizations, respectively. It is obvious, that this expression coincides formally with the analogical one-loop contribution to the normalization condition of the hadron wave function (see for example, (90)). Nevertheless there is an essential difference in the interpretation of two results. Namely, in (63) the wave function Ψ has the small energy propagator $1/\Delta E$ for the final partons whereas in (148) we have such propagator for the initial parton. The difference is especially obvious if one considers the wave functions for the state containing several ($n > 2$) final particles with their momenta $k_1, k_2, ..., k_n$ and the Sudakov components $x_1, x_2, ..., x_n$. In this case for the intermediate states having r partons with momenta $q_1, q_2, ..., q_r$ the hadron wave function has the energy propagator

$$(m^2 - \sum_{k=1}^{r} \frac{q_k^2}{\beta_k})^{-1} , \tag{150}$$

but for the parton wave function the corresponding propagator is

$$\left(\sum_{i=1}^{n} \frac{k_i^2}{\beta_i} - \sum_{k=1}^{r} \frac{q_k^2}{\beta_k}\right)^{-1} . \tag{151}$$

As it was mentioned above, the second important peculiarity in the case of fragmentation functions in comparison with parton distributions is related to the kinematics for the flow of particle transverse momenta. In the parton shower drawn in Fig.(9) for the hadron wave

function each pair of partons is produced with significantly larger transverse momenta than the transverse momentum of the initial particle, but in the case of the transition of a parton to hadrons each produced pair of partons (j,l) flies almost in the same direction as the initial parton (i) and has much smaller relative transverse momentum. To show it, let us consider the contribution of this elementary transition $i \to j,l$ to the denominator of the energy propagator (151) and impose on momenta the condition of its relative smallness

$$\frac{k_i^2}{\beta_i} - \frac{k_j^2}{\beta_j} - \frac{k_l^2}{\beta_l} \ll k_i^2 \,. \tag{152}$$

It is obvious, that the left hand side is zero for the planar kinematics when the emission angles $\theta \sim k_\perp/\beta$ for their momenta are the same

$$\frac{k_i}{\beta_i} = \frac{k_j}{\beta_j} = \frac{k_l}{\beta_l} \,. \tag{153}$$

We obtain the logarithmic contribution over the transverse momenta of produced particles i,j providing that

$$\left(\frac{k_j}{\beta_j} - \frac{k_i}{\beta_i}\right)^2 \ll \left(\frac{k_i}{\beta_i}\right)^2 \,. \tag{154}$$

It means, that the relative angles $\Delta\theta_i$ of produced pairs decrease in the course of the subsequent decays

$$1 \gg \Delta\theta_1 \gg \Delta\theta_2 \gg ... \gg \Delta\theta_r \,. \tag{155}$$

Thus, the hadron and parton wave functions are different even on the partonic level, when instead of final hadrons h we consider the partons with a fixed virtuality $k^2 \sim \Lambda^2_{QCD}$. Nevertheless for the last case, in the region of fixed β the evolution equations for the parton distributions $D_h^k(x)$ and for the fragmentation functions $\bar{D}_h^k(x)$ in the leading logarithmic approximation coincide and one can obtain the Gribov-Lipatov relation [20, 21]

$$\bar{D}_h^k(x) = D_h^k(x) \,, \tag{156}$$

which illustrates a duality between the theoretical descriptions of hard processes in terms of hadrons and partons. In the next-to-leading approximation this relation is violated, because in the fragmentation function at small x there appear double-logarithmic contributions $\sim \alpha^2 \ln^3 x \, \ln Q^2$, which are absent in the parton distributions.

Another interesting equality valid in the Born approximation is the so-called Drell-Levy-Yan relation [12]

$$\bar{D}_k^h(x) = (-1)^{2(s_k - s_h)+1} \, x \, D_h^k\left(\frac{1}{x}\right) \,. \tag{157}$$

where s_h and s_k are the spins of the corresponding particles. This relation is violated in QED and QCD already in the leading logarithmic approximation, because the point $x = 1$ turns out to be a singular point $D_h^k(x) \sim (1-x)^a$ with a non-integer value of a. In each order of the perturbation theory we have a polynomial of logarithms $\ln(1-x)$ with the coefficients which are analytic functions of x. The receipt for the analytic continuation of $D_h^k(x)$ around the point $x = 1$ compatible with the relation (157) is simple: the arguments of logarithms should be taken as the modules $|1-x|$ at $x > 1$ with an analytic continuation of the coefficients of the polynomials [22].

7.6 Parton distributions in QCD in LLA

To solve the evolution equations (76) for the parton distributions $n_k(x)$ in a leading logarithmic approximation of QCD one can use the scale invariance of their integral kernels to the dilatations $x \to \lambda\, x$ in the Bjorken variable. This invariance is related to the conservation of the total angular momentum j in the crossing channel t. Thus, we search the solution $n_k(x)$ of the DGLAP equations for QCD in the form of the Mellin representation in the variable $\ln(1/x)$

$$n_k(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j n_k^j \,, \tag{158}$$

where the analytically continued moments n_k^j of the parton distributions

$$n_k^j = \int_0^1 dx\, x^{j-1}\, n_k(x) \tag{159}$$

in LLA obey the system of the ordinary differential equations

$$\frac{\partial}{\partial \xi}\, n_k^j = \sum_r w_{r\to k}^j\, n_r^j \,. \tag{160}$$

Here the variable ξ is defined in eq. (79) and (80) as follows

$$\xi(Q^2) = \frac{2N_c}{\beta_s} \ln \frac{\alpha_\mu}{\alpha(Q^2)} \tag{161}$$

and the elements of the anomalous dimension matrix $w_{r\to k}^j$ are calculated above. This matrix has the block-diagonal form in an appropriate basis and can be easily diagonalized

$$\sum_r w_{r\to k}^j\, n_r^j(s) = w^j(s)\, n_k^j(s) \,. \tag{162}$$

Its eigenvalues $w^j(s)$ and eigenfunctions $n_k^j(s)$ describe the asymptotic behavior of the multiplicatively renormalized matrix elements of the twist-two operators O_s^j

$$n_k^j(s, Q^2) = < h|O_s^j|h > \tag{163}$$

at a large ultraviolet cut-off $\Lambda^2 = Q^2$

$$n_k^j(s, Q^2) = n_k^j(s, Q^2) \exp\Big(\Delta\xi\, w^j(s)\Big) \;,\; \Delta\xi = \xi(Q^2) - \xi(Q_0^2) \,. \tag{164}$$

Note, that the asymptotic freedom, corresponding to a logarithmic vanishing of $\alpha(Q^2)$ at large Q^2, leads to a rather weak dependence of $n_k^j(s)$ from Q^2. The value of the matrix element $n_k^j(s)$ at $Q^2 = Q_0^2$ is an initial condition for the evolution equation and should be extracted from the experimental data.

The moments of the parton distributions n_k^j are linear combinations of the eigenfunctions $n_k^j(s, Q^2)$ (164). Partly the coefficients in these combinations are fixed by the quantum numbers of the corresponding twist-2 operators. But there are operators having the same quantum numbers and constructed from the gluon or quark fields. They are mixed each with another in the process of the renormalization. The linear combinations of the parton distributions which are multiplicatively renormalizable are obtained from the diagonalization of the anomalous dimension matrix $w_{r\to k}^j$ calculated above.

We write below the moments n_k^j entering in expression (158) for the parton distributions $n_k(x)$ being solutions of the DGLAP equations (76) in LLA. The helicity of the initial hadron with its spin s is assumed to be $+s$. The simplest result is obtained for momenta of the non-singlet distributions

$$\frac{1}{2}\left(n_q(j)-n_{\bar{q}}(j)\right)_{Q^2}=\frac{1}{2}\left(n_q(j)-n_{\bar{q}}(j)\right)_{Q_0^2}\exp\left(\Delta\xi\, w^j_{1/2^+\to 1/2^+}\right)\,, \tag{165}$$

where the helicity is conserved $\lambda_h=\lambda_q$ and due to (131) the anomalous dimension $w^j_{1/2^+\to 1/2^+}$ is

$$w^j_{1/2^+\to 1/2^+}=\frac{N_c^2-1}{2N_c^2}\left(2\psi(1)-2\psi(j)-\frac{1}{j+1}-\frac{1}{j}+\frac{3}{2}\right)\,. \tag{166}$$

For the singlet flavor quantum numbers in the crossing channel the corresponding expressions are more complicated. We begin with the distributions of non-polarized quarks and gluons ($i=1/2,1$)

$$\frac{1}{2}\left(n_{i+}(j)+n_{i-}(j)\right)=\sum_{s=1,2}V_s^i(j)\exp\left(\Delta\xi\, w_s^{jv}\right)\,. \tag{167}$$

The anomalous dimension matrix for the non-polarized case was calculated above (see expressions (131), (132), (122) and (106))

$$\begin{aligned}
w^{jv}_{1/2\to 1/2}&=\frac{N_c^2-1}{2N_c^2}\left(2\psi(1)-2\psi(j)-\frac{1}{j+1}-\frac{1}{j}+\frac{3}{2}\right)\,,\\
w^{jv}_{1/2\to 1}&=\frac{N_c^2-1}{2N_c^2}\left(\frac{2}{j-1}-\frac{2}{j}+\frac{1}{j+1}\right)\,,\\
w^{jv}_{1\to 1/2}&=\frac{n_f}{N_c}\left(\frac{1}{j}-\frac{2}{j+1}+\frac{2}{j+2}\right)\,,\\
w^{jv}_{1\to 1}&=2\psi(1)-2\psi(j-1)-\frac{2}{j+2}+\frac{2}{j+1}-\frac{4}{j}+\frac{11}{6}-\frac{n_f}{3N_c}\,.
\end{aligned} \tag{168}$$

The coefficients $V_s^i(j)$ and quantities w_s^{jv} for $s=1,2$ are respectively two eigenfunctions and eigenvalues of the anomalous dimension matrix $w^{ja}_{k\to i}$

$$w^{jv}\,V_i(j)=\sum_{k=1,1/2}w^{jv}_{k\to i}\,V_k(j) \tag{169}$$

with the initial condition

$$\sum_{r=1,2}V_i^r(j)=\frac{1}{2}\left(n_{i+}(j)+n_{i-}(j)\right)_{|_{Q^2=Q_0^2}} \tag{170}$$

fixed by experimental data at $Q^2=Q_0^2$.

In a similar way one can construct the solution of the evolution equations (76) describing the Q^2-dependence of the difference of distributions for the partons having the same and opposite helicities with respect to that of the initial hadron

$$\frac{1}{2}\left(n_{i+}(j)-n_{i-}(j)\right)=\sum_{s=1,2}A_i^s(j)\exp\left(\Delta\xi\, w_s^{ja}\right) \tag{171}$$

with the use of the anomalous dimension matrix for the polarized particles (see (131), (133)), (123) and (107)

$$w^{ja}_{1/2^+\to 1/2^+} = \frac{N_c^2-1}{2N_c^2}\left(2\psi(1)-2\psi(j)-\frac{1}{j+1}-\frac{1}{j}+\frac{3}{2}\right),$$

$$w^{ja}_{1/2\to 1} = \frac{N_c^2-1}{2N_c^2}\left(\frac{2}{j}-\frac{1}{j+1}\right).$$

$$w^{ja}_{1\to 1/2}(x) = \frac{n_f}{N_c}\left(-\frac{1}{j}+\frac{2}{j+1}\right),$$

$$w^{ja}_{1\to 1} = 2\psi(1)-2\psi(j)-\frac{4}{j+1}+\frac{2}{j}+\frac{11}{6}-\frac{n_f}{3N_c}. \tag{172}$$

In this case the coefficients $A_i^s(j)$ and quantities w_s^{ja} are obtained from the eigenvalue equation for the above matrix $w^j_{k\to i}$

$$w^{ja}\,A_i(j) = \sum_{k=1,1/2} w^{ja}_{k\to i}\,A_k(j) \tag{173}$$

with the additional condition

$$\sum_{s=1,2} A_i^s(j) = \frac{1}{2}\left(n_{i+}(j)-n_{i-}(j)\right)_{|_{Q^2=Q_0^2}} \tag{174}$$

fixed by experimental data at $Q^2=Q_0^2$.

One can assume, for example, that already for $Q^2=Q_0^2$ the formulas of the parton model are correct. Than for $Q^2\gg Q_0^2$ we have

$$n_k(x,Q^2) = \sum_r \int_x^1 \frac{dy}{y}\, n_r(y,Q_0^2)\, W_{r\to k}(x/y), \tag{175}$$

where $W_{r\to k}(x)$ is an inclusive probability to find the hard parton k inside a comparatively soft parton r and $n_r(y,Q_0^2)$ is a parton distribution function at the scale Q_0^2.

The inclusive probabilities $W_{r\to k}(x)$ are normalized as follows

$$W_{r\to k}(x)_{|_{Q^2=Q_0^2}} = \delta_{r,k}\,\delta(x-1). \tag{176}$$

Their momenta

$$W_{r\to k}(j) = \int_0^1 dx\, x^{j-1}\, W_{r\to k}(x) \tag{177}$$

can be calculated in LLA

$$\frac{1}{2}\left(W_{q\to q}(j)-W_{q\to\bar q}(j)\right) = \frac{1}{2}\left(W_{\bar q\to\bar q}(j)-W_{q\to\bar q}(j)\right) = \exp\left(\Delta\xi\, w^j_{1/2^+\to 1/2^+}\right) \tag{178}$$

and

$$\frac{1}{2}\left(W_{r+\to i+}(j)+W_{r+\to i-}(j)\right) = \sum_{s=\pm} V^s_{r\to i}(j)\exp\left(\Delta\xi\, w_s^{jv}\right), \tag{179}$$

$$\frac{1}{2}\left(W_{r+\to i+}(j)-W_{r+\to i-}(j)\right) = \sum_{s=\pm} A^s_{r\to i}(j)\exp\left(\Delta\xi\, w_s^{ja}\right), \tag{180}$$

where

$$w_{\pm}^{jv} = \frac{w_{1/2\to1/2}^{jv} + w_{1\to1}^{jv}}{2} \pm \sqrt{\frac{1}{4}\left(w_{1/2\to1/2}^{jv} - w_{1\to1}^{jv}\right)^2 + w_{1/2\to1}^{jv} w_{1\to1/2}^{jv}}\,,$$

$$w_{\pm}^{ja} = \frac{w_{1/2\to1/2}^{ja} + w_{1\to1}^{ja}}{2} \pm \sqrt{\frac{1}{4}\left(w_{1/2\to1/2}^{ja} - w_{1\to1}^{ja}\right)^2 + w_{1/2\to1}^{ja}\, w_{1\to1/2}^{ja}}\,, \tag{181}$$

$$V_{q\to q}^{\pm}(j) = -\frac{w_{\mp}^{jv} - w_{1/2\to1/2}^{jv}}{w_{\pm}^{jv} - w_{\mp}^{jv}}\,,\ V_{g\to q}^{\pm}(j) = \frac{w_{1\to1/2}^{jv}}{w_{\pm}^{jv} - w_{\mp}^{jv}}\,,$$

$$V_{q\to g}^{\pm}(j) = \frac{w_{1/2\to1}^{jv}}{w_{\pm}^{jv} - w_{\mp}^{jv}}\,,\ V_{g\to g}^{\pm}(j) = \frac{w_{\pm}^{jv} - w_{1/2\to1/2}^{jv}}{w_{\pm}^{jv} - w_{\mp}^{jv}}\,, \tag{182}$$

$$A_{q\to q}^{\pm}(j) = -\frac{w_{\mp}^{ja} - w_{1/2\to1/2}^{ja}}{w_{\pm}^{ja} - w_{\mp}^{ja}}\,,\ A_{g\to q}^{\pm}(j) = \frac{w_{1\to1/2}^{ja}}{w_{\pm}^{ja} - w_{\mp}^{ja}}\,,$$

$$A_{q\to g}^{\pm}(j) = \frac{w_{1/2\to1}^{ja}}{w_{\pm}^{ja} - w_{\mp}^{ja}}\,,\ A_{g\to g}^{\pm}(j) = \frac{w_{\pm}^{ja} - w_{1/2\to1/2}^{ja}}{w_{\pm}^{ja} - w_{\mp}^{ja}}\,. \tag{183}$$

In particular, we have

$$\frac{1}{2}\left(W_{q\to q}(1) - W_{q\to\bar{q}}(1)\right) = \frac{1}{2}\left(W_{\bar{q}\to\bar{q}}(1) - W_{q\to\bar{q}}(1)\right) = 1\,, \tag{184}$$

which corresponds to the barion number conservation. Further, for momenta of the non-polarized parton distributions with $j = 2$ one can obtain

$$W_{q\to q}(2) = \frac{w_{+}^{2v} - w_{1/2\to1/2}^{2v}}{w_{+}^{2v}} + \frac{w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\,\exp\left(\Delta\xi\, w_{+}^{2v}\right)\,,$$

$$W_{q\to g}(2) = \frac{w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\left(1 - \exp\left(\Delta\xi\, w_{+}^{2v}\right)\right)\,,$$

$$W_{g\to q}(2) = \frac{w_{+}^{2v} - w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\left(1 + \exp\left(\Delta\xi\, w_{+}^{2v}\right)\right)\,,$$

$$W_{g\to g}(2) = \frac{w_{1/2\to1/2}^{2v}}{w_{+}^{2v}} + \frac{w_{+}^{2v} - w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\,\exp\left(\Delta\xi\, w_{+}^{2v}\right)\,, \tag{185}$$

where $w_{-}^{2v} = 0$, because the energy-momentum tensor $\theta_{\mu\nu}(x)$ is not renormalized. From these expressions we obtain the sum rule

$$W_{h\to g}(2) + W_{h\to q}(2) = 1 \tag{186}$$

valid for any initial particle h, which corresponds to the parton energy conservation in the hadron infinite momentum frame $\vec{p} \to \infty$. Moreover, at $\Delta\xi \to \infty$, where the contribution $\exp\left(\Delta\xi\, w_{+}^{2v}\right)$ tends to zero, we obtain, that the average energy taken by the gluon or quark does not depend on the type of the initial particle h

$$W_{h\to g}(2) = \frac{w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\,,\ W_{h\to q}(2) = \frac{w_{+}^{2v} - w_{1/2\to1/2}^{2v}}{w_{+}^{2v}}\,. \tag{187}$$

This property is a consequence of the factorization of the parton distribution momenta $D_k^l(j)$ for each multiplicatively renormalized operator

$$D_k^l(j) = \sum_{r=1,2} b_k(r) b^l(r) \exp\left(\Delta\xi\, w^j\right) .$$

For the quasi-elastic region $x \to 1$ the large values of the Lorentz spin $j \sim 1/(1-x)$ are essential in the Mellin representation for $n_k(x)$. In the limit $j \to \infty$ the anomalous dimensions are simplified

$$w_+^j \approx w_{1/2\to 1/2}^j \approx w_+^{jv} \approx w_+^{ja} \approx \frac{N_c^2-1}{2N_c^2}\left(-2\ln j + \frac{3}{2}\right) ,$$

$$w_-^j \approx w_-^{jv} \approx w_-^{ja} \approx -2\ln j + \frac{11}{6} - \frac{n_f}{3N_c} . \tag{188}$$

Using also eqs. (182) and (183) one can verify, that in the quasi-elastic regime the inclusive probability $W_{q\to g}(x)$ is small and in other transitions the helicity sign is conserved leading to the relation $A \approx V$. The momenta of non-vanishing inclusive probabilities at large j are

$$W_{q+\to g+}(j) \approx \frac{1}{2}\frac{N_c^2-1}{N_c^2+1}\frac{1}{j\ln j}\left(e^{\Delta\xi\, w_+^j} - e^{\Delta\xi\, w_-^j}\right) ,\ W_{g+\to q+}(j) \approx \frac{2\, n_f\, N_c}{N_c^2-1} W_{q+\to g+}(j) ,$$

$$V_{q+\to q+}(j) \approx e^{\Delta\xi\, w_+^j} ,\ V_{g+\to g+}(j) \approx e^{\Delta\xi\, w_-^j} . \tag{189}$$

Similar results are valid for transitions between the partons with negative helicities.

Inserting the above expressions in the Mellin integral representation (158) we can obtain the following behavior of inclusive probabilities

$$W_{g+\to q+}(j) \approx \frac{2\, n_f\, N_c}{N_c^2-1} V_{q+\to g+}(j) ,\ W_{q+\to g+}(x)_{|x\to 1} \approx$$

$$\frac{1}{2}\frac{N_c^2-1}{N_c^2+1}\left(\frac{\exp\left(\frac{3}{4}\frac{N_c^2-1}{N_c^2}\Delta\xi\right)}{\Gamma(1+\frac{N_c^2-1}{N_c^2}\Delta\xi)}\frac{(1-x)^{\frac{N_c^2-1}{N_c^2}\Delta\xi}}{\ln\frac{1}{1-x}} - \frac{\exp\left((\frac{11}{6}-\frac{n_f}{3N_c})\Delta\xi\right)}{\Gamma(1+2\Delta\xi)}\frac{(1-x)^{2\Delta\xi}}{\ln\frac{1}{1-x}}\right) ,$$

$$W_{q+\to q+}(x)_{|x\to 1} \approx \frac{\exp\left(\frac{3}{4}\frac{N_c^2-1}{N_c^2}\Delta\xi\right)}{\Gamma(\frac{N_c^2-1}{N_c^2}\Delta\xi)}(1-x)^{-1+\frac{N_c^2-1}{N_c^2}\Delta\xi} ,$$

$$W_{g+\to g+}(x)_{|x\to 1} \approx \frac{\exp\left((\frac{11}{6}-\frac{n_f}{3N_c})\Delta\xi\right)}{\Gamma(2\Delta\xi)}(1-x)^{-1+2\Delta\xi} . \tag{190}$$

Note, that the expressions for $W_{q+\to q+}(x)_{|x\to 1}$ and $W_{g+\to g+}(x)_{|x\to 1}$ contain the sum of the Sudakov double-logarithmic terms (cf. (32)).

Let us consider now the small-x behavior of inclusive probabilities $W_{g+\to g+}(x)$. It is related to the singularities of the anomalous dimension matrix w (167), (168) and (172) at $j \to 1$. Only two elements of this matrix are singular

$$\lim_{j\to 1} w_{1/2\to 1}^{jv}(x) = \frac{N_c^2-1}{2N_c^2}\left(\frac{2}{j-1} - \frac{3}{2}\right) ,\ \lim_{j\to 1} w_{1\to 1}^{jv} = \frac{2}{j-1} - \frac{11}{6} - \frac{n_f}{3N_c} , \tag{191}$$

which leads to the singular eigenvalue

$$w_+^{jv} \approx \frac{2}{j-1} - \frac{11}{12} + \frac{n_f}{6N_c}\,\frac{N_c^2-2}{N_c^2} \tag{192}$$

entering in the following non-vanishing momenta $W_{r\to k}(j)$ (179)

$$\lim_{j\to 1} W_{q\to g}(j) \approx \frac{N_c^2-1}{2N_c}\,\exp(\Delta\xi\, w_+^{jv})\ ,\ \lim_{j\to 1} W_{g\to g}(x) \approx \exp(\Delta\xi\, w_+^{jv})\,. \tag{193}$$

For $x \to 0$ the Mellin integral (158) has a saddle point situated at

$$j-1 = \sqrt{\frac{2\,\Delta\xi}{\ln\frac{1}{x}}}\,. \tag{194}$$

The calculation of the integral by the saddle point method gives the following result for the inclusive probabilities

$$W_{q\to g}(x) \approx \frac{N_c^2-1}{2N_c}\,W_{g\to g}(x)\,,$$

$$W_{g\to g}(x) \approx \frac{1}{x}\,\exp\left(2\sqrt{2\Delta\xi\,\ln\frac{1}{x}}\right)\left(\frac{\Delta\xi}{8\,\pi^2\,\ln^3\frac{1}{x}}\right)^{1/4}\exp\left(\left(-\frac{11}{12}+\frac{n_f}{6N_c}\,\frac{N_c^2-2}{N_c^2}\right)\,\Delta\xi\right)\,. \tag{195}$$

Thus, the gluon distributions grow rapidly with increasing $\Delta\xi$ and decreasing x. It turns out, that by gathering the powers of $g^2\ln 1/x$ in in all orders of perturbation theory in the framework of the BFKL approach one can obtain an even more significant increase of these distributions [4].

Because the leading singularity of the anomalous dimension $w^j_{1/2\to 1/2}$ (167) of the non-singlet quark distribution is situated at $j=0$

$$\lim_{j\to 0} w^j_{1/2\to 1/2} \approx \frac{N_c^2-1}{2N_c^2}\left(\frac{1}{j}-\frac{1}{2}\right) \tag{196}$$

the small-x behavior of this distribution is less singular

$$\lim_{x\to 0}\frac{1}{2}\left(W_{q\to q}(x) - W_{q\to\bar{q}}(x)\right) \sim \exp\sqrt{\frac{2\Delta\xi\,\ln\frac{1}{x}\,(N_c^2-1)}{N_c^2}}\,. \tag{197}$$

7.7 Parton number correlators

The knowledge of the parton wave function gives us a possibility to calculate more complicated quantities - the parton correlators. The simplest generalization of the parton number distribution is the parton number correlator having the representation

$$< h|n_{l_1}n_{l_2}...n_{l_k}|h >= Z_h\prod_{r=1}^{k}\delta_{l_r h}$$

$$+\sum_{n=2}^{\infty}\sum_{i_1...i_n} n_{l_1}n_{l_2}...n_{l_k}\int\prod_{i=1}^{n}\frac{d^2k_{\perp i}}{(2\pi)^3}\,\frac{d\beta_i}{\beta_i}\theta(\beta_i)Z_g^{n_g}Z_q^{n_q}Z_{\bar{q}}^{n_{\bar{q}}}|\psi|^2(2\pi)^3\delta^2(\sum_i k_{\perp i})\delta(1-\sum_i\beta_i)\,, \tag{198}$$

where $Z_s^{1/2}$ $(s = g, q, \bar{q})$ are the renormalization constants for the corresponding wave functions.

By differentiating this expression over the variable ξ (79) related to the ultraviolet cut-off for the transverse momenta of partons and taking into account that in the essential region of integration the parton transverse momenta rapidly grow after each partonic decay, we obtain the equation for the parton number correlators

$$\frac{d}{d\xi} < h| \prod_{r=1}^{k} n_{l_r} |h >= -\sum_s W_s \; < h| n_s \prod_{r=1}^{k} n_{l_r} |h >$$

$$+ \sum_s \sum_{r=1}^{k} \sum_{i_1, i_2} W_{s \to (i_1, i_2)} < h| n_s \prod_{r=1}^{k} \left(n_{l_r} - \delta_{l_r s} + \delta_{l_r i_1} + \delta_{l_r i_2} \right) |h > \tag{199}$$

with the initial conditions

$$< h| \prod_{r=1}^{k} n_{l_r} |h >_{|\xi=0} = \prod_{r=1}^{k} \delta_{l_r h} \, . \tag{200}$$

In the right hand side of this equation the terms containing $(k+1)$ factors of n_l are cancelled due to the relation

$$W_s = \sum_{i_1, i_2} W_{s \to (i_1, i_2)} \tag{201}$$

and therefore it relates the parton number correlators of the k-th order with low order correlators which can be considered as inhomogeneous contributions.

To illustrate the essential features of the solution of such equation we neglect in it the quark and anti-quark contributions considering the pure Yang-Mills model. Further, we consider only the total number of gluons with two helicities

$$n = n_+ + n_- \, . \tag{202}$$

Introducing the following function of the gluon numbers

$$T^{(k)}(n) = n(n+1)...(n+k-1) \tag{203}$$

one can verify from (199), that its matrix elements satisfy the linear evolution equation

$$\frac{d}{d\xi} < h|T^{(k)}(n)|h >= W \, k \; < h|T^{(k)}(n)|h > , \quad < h|T^{(k)}(n)|h >_{|\xi=0} = k! \, . \tag{204}$$

Its solution is simple

$$< h|T^{(k)}(n)|h >= k! \, e^{k \, W \, \xi} \, . \tag{205}$$

Further, the probability to find a hadron in the state with the n partons is

$$P_n = \sum_{i_1 \ldots i_n} \int \prod_{i=1}^{n} \frac{d^2 k_{\perp i}}{(2\pi)^3} \frac{d\beta_i}{\beta_i} \theta(\beta_i) Z_g^{n_g} Z_q^{n_q} Z_{\bar{q}}^{n_{\bar{q}}} |\psi|^2 (2\pi)^3 \delta^2 (\sum_i k_{\perp i}) \delta(1 - \sum_i \beta_i) \, , \tag{206}$$

where in particular (for h coinciding with one of partons)

$$P_1 = Z_h \, . \tag{207}$$

It is obvious, that due to the normalization condition for the wave function together with eq (204) W_n for the pure gluonic case satisfies the sum rules

$$\sum_{n=1}^{\infty} P_n = 1\,, \ \sum_{n=1}^{\infty} T^{(k)}(n)\, P_n = \bar{n}^k = e^{k\,W\,\xi} \tag{208}$$

and the evolution equation

$$\frac{d}{d\xi} P_n = W\left(-n\, P_n + (n-1)\, P_{n-1}\right)\,, \ (P_n)_{|\xi=0} = \delta_{n,1}\,. \tag{209}$$

In an explicit form its solution written below

$$P_n = \frac{1}{\bar{n}} \left(\frac{\bar{n}-1}{\bar{n}}\right)^{n-1}\,, \ \bar{n} = e^{W\,\xi} \tag{210}$$

is different from the Poisson distribution

$$P_n^P = \frac{1}{n!}\,(\bar{n})^n\, e^{-\bar{n}}\,. \tag{211}$$

Note, that P_n satisfies the sum rules (208) and for $\bar{n} \to \infty$ it has the property of the KNO-scaling

$$\lim_{\bar{n}\to\infty} P_n = \frac{1}{\bar{n}}\, f\left(\frac{n}{\bar{n}}\right)\,, \ f(x) = e^{-x}\,. \tag{212}$$

In the real QCD case we can generalize the above discussion by introducing the probability $P(n_g^+, n_g^-, n_q^+, n_q^-, n_{\bar{q}}^+, n_{\bar{q}}^-)$ to find a hadron in the states with a certain number n_t of partons $(t = g^\pm, q^\pm, \bar{q}^\pm)$ with definite helicities $\lambda = \pm s$. This probability satisfies the equation

$$\frac{d}{d\xi} P(n_t) = -\sum_r n_r\, W_r\, P(n_t) + \sum_r \sum_{i,j} (n_r + 1 - \delta_{r,i} - \delta_{r,j})\; W_{r\to(i,j)}\; P\left(n_t + \delta_{t,r} - \delta_{t,i} - \delta_{t,j}\right)\,. \tag{213}$$

In the second term of the right hand side of this equation $\sum_t n_t$ is diminished by unity and therefore after the Mellin transformation

$$P(n_t) = \int_{-i\infty}^{\infty} \frac{d\nu}{2\pi i}\, e^{\nu\xi}\, P_\nu(n_t) \tag{214}$$

one can obtain a recurrent relation for $P_\nu(n_t)$.

It is natural to introduce the more informative quantity for the description of the hadron structure–the exclusive distributions of partons as functions of their Feynman parameters β_r^t $(r = 1, 2, ...n_t)$, where t is the sort of a parton (including its helicity)

$$f_{n_t}(\beta_r^t) = \frac{1}{\prod_t n_t!} \sum_{i_1 \ldots i_n} \int \prod_{t,l} \frac{d^2 k_{\perp l}^t}{(2\pi)^3\, \beta_l^t} \prod_t Z_t^{n_t} |\psi_h(k_{\perp l}^t, \beta_l^t)|^2\, (2\pi)^3\, \delta(\sum k_{\perp l}^t)\, \delta(1 - \sum \beta_l^t)\,. \tag{215}$$

This quantity satisfies the following evolution equation

$$\frac{d}{d\xi} f_{n_t}(\beta_r^t) = -\sum_s n_s W_s f(\beta_r^s)$$

$$+ \sum_{s,m} \sum_{i,j} W_{s\to(i,j)}(\beta_m^s, \beta_{l_1}^i, \beta_{l_2}^j)\, \delta(\beta_m^s - \beta_{l_1}^i - \beta_{l_2}^j))\, f(n_t + \delta_{ts} - \delta_{ti} - \delta_{tj})(\beta_l^t)\, d\,\beta_m^s \tag{216}$$

with the initial conditions

$$f_{n_t}(\beta_r^t)_{|\xi=0} = \delta_{n_t 1} \prod_{i\neq 1} \delta_{n_i 0} \delta(\beta^n - 1)\,, \tag{217}$$

One can define the generating functional for these exclusive distributions

$$I(\phi_t) = \sum_{n_t} \int \prod_t \prod_{l=1}^{n_t} \phi_t(\beta_l^t)\, d\beta_l^t\, f_{n_t}(\beta_r^t)\,, \tag{218}$$

where $\phi_t(\beta)$ $(t = g_\pm, q_\pm, \bar{q}_\pm)$ are some auxiliary fields. The evolution equation for $f_{n_t}(\beta_r^t)$ is equivalent to the following equation in the variational derivatives $\delta/(\delta\phi_s(x))$ for the functional $I(\phi_t)$

$$\frac{d}{d\xi} I = -\sum_s W_s \int \phi_s(x) \frac{\delta I}{\delta \phi_s(x)} d\,x$$

$$+ \sum_s \sum_i \sum_j \int dx_1\, dx_2\, W_{s\to(i,j)}(x_1 + x_2, x_1, x_2)\, \phi_i(x_1) \phi_j(x_2)\, \frac{\delta I}{\delta \phi_s(x_1 + x_2)} \tag{219}$$

with the initial condition

$$I_{|\xi=0} = \phi_h(1)\,. \tag{220}$$

Exclusive distributions can be calculated by taking the functional derivative from I

$$f_{n_t}(\beta_l^t) = \frac{1}{\prod_t n_t!} \prod_t \prod_l \frac{\delta I}{\delta \phi_t(\beta_l^t)}_{|\phi_t(\beta)=0}\,. \tag{221}$$

Moreover, we can calculate from this functional also inclusive correlators of partons having the Feynman parameters β_l^t

$$D_{n_t}(\beta_l^t) = \frac{1}{\prod_t n_t!} \prod_t \prod_l \frac{\delta I}{\delta \phi_t(\beta_l^t)}_{|\phi_t(\beta)=1}\,. \tag{222}$$

It is related with the fact, that the generating functional J for the inclusive correlators is expressed in terms of I

$$J(\phi_t) = \sum_{n_t} \int \prod_t \prod_{l=1}^{n_t} d\beta_l^t D_{n_t}(\beta_l^t)\,, \tag{223}$$

related to the functional I as follows

$$J(\phi_t) = I(1 + \phi_t)\,. \tag{224}$$

Due to the normalization condition for the wave function the functionals I and J are normalized as follows

$$I(1) = J(0) = 1\,. \tag{225}$$

The energy momentum conservation $\sum_r \beta_r = 1$ imposes the following constraint for the functional I

$$I(e^{\lambda\beta}\phi(\beta)) = e^\lambda I(\phi(\beta))\,. \tag{226}$$

The generating functional for the inclusive probabilities satisfies the following equation

$$\frac{d}{d\xi} J = -\sum_s W_s \int \phi_s(x) \frac{\delta J}{\delta \phi_s(x)} d\,x +$$

$$\sum_s \sum_i \sum_j \int dx_1\, dx_2\, W_{s\to(i,j)}(x_1+x_2,x_1,x_2)\ (\phi_i(x_1)+\phi_j(x_2)+\phi_i(x_1)\phi_j(x_2))\ \frac{\delta J}{\delta\phi_s(x_1+x_2)} \tag{227}$$

with the initial condition

$$J(\phi_t)_{|\xi=0} = 1+\phi_h(1)\,. \tag{228}$$

The inclusive distributions can be obtained from J by the functional differentiation

$$D_{n_t}(\beta_l^t) = \frac{1}{\prod_t n_t!}\,\frac{\delta J}{\delta\phi_t(\beta_l^t)}_{|\phi_t(\beta)=0}\,. \tag{229}$$

From the above expressions we can obtain the evolution equation for the functions $D_{n_t}(\beta_l^t)$. It has a recurrence form and allows one to calculate them in terms of the simple inclusive probabilities $D_h^i(\beta)$.

For the solution of the evolution equation for the functional I we can use the simple mathematical trick and consider the so-called characteristic equation

$$\frac{d}{d\xi}\,\phi_s(x,\xi) = -W_s\,\phi_s(x,\xi)$$

$$+\sum_{k,r}\int dx_1\, dx_2\,\phi_k(x_1,\xi)\,\phi_r(x_2,\xi)\,\delta(x_1+x_2-x)\,W_{s\to(k,r)}(x,x_1,x_2)\,. \tag{230}$$

Let us assume, that for any function $\phi_k(x,\xi)$ one can calculate the initial condition

$$\phi_k(x,0) = \chi_k\,(\phi_r(x,\xi),\ x)\ , \tag{231}$$

where χ_s are functions of x and functionals from $\phi_r(x,\xi)$. Then it is easily verify, that the solution of the evolution equation for I can be constructed as follows

$$I = \chi_h\,(\phi_r(x,\xi),\ 1)\ . \tag{232}$$

Note, that this expression satisfies the initial condition.

To illustrate this mathematical method we consider the pure Yang-Mills theory and only the most singular terms in the splitting kernels

$$W_{s\to(i,k)}(x,x_1,x_2) = 2\,\frac{x}{x_1\,x_2}\,,\ \ W = \int_0^1 \frac{d\,x}{x(1-x)}\,, \tag{233}$$

which corresponds to the $N=4$ extended super-symmetric theory where the anomalous dimension is universal for all twist-2 operators

$$w_j = 2\int_0^1 \frac{d\,x}{x\,(1-x)}\left(x^{j-1}-x\right) = 2\psi(1)-2\psi(j-1)\,. \tag{234}$$

In this case the characteristic equation takes the form

$$\frac{d}{d\xi}\,\phi(x,\xi) = -W\,\phi(x,\xi) + 2\int \frac{x\,dx_1\,dx_2}{x_1\,x_2}\,\delta(x_1+x_2-x)\,\phi(x_1,\xi)\,\phi(x_2,\xi)\,. \tag{235}$$

For simplicity we disregard complications related to infrared divergencies of integrals and consider W as a constant. Then the solution can be searched in the form of the Mellin transformation

$$\chi(p,\xi) = \int_0^\infty \frac{dx}{x}\,e^{-px}\,\phi(x,\xi)\,, \tag{236}$$

where $\phi(x,\xi)$ satisfies the equation

$$\frac{d}{d\xi}\,\chi(p,\xi) = -W\,\chi(p,\xi) + 2\,\chi^2(p,\xi)\,. \tag{237}$$

Therefore we can find $\chi(p,\xi)$ for any initial condition $\chi(p,0)$

$$\chi(p,\xi) = \frac{W\,\chi(p,0)\,e^{-W\,\xi}}{W - 2\,\chi(p,0)\,(1-e^{-W\,\xi})}\,, \tag{238}$$

which allows us to calculate the generating functional I

$$I(\phi) = \int_{-i\infty}^{i\infty}\frac{d\,p}{2\pi i}\,e^p\,\frac{W\,\int_0^\infty\,e^{-xp}\,\frac{d\,x}{x}\,\phi(x,\xi)}{2\,(1-\exp(-W\xi))\int_0^\infty\,e^{-xp}\,\frac{dx}{x}\,\phi(x,\xi) + W\,\exp(-W\xi)}\,. \tag{239}$$

Note, however, that for self-consistency of the approach one should regularize the splitting kernel in the infrared region for example by introducing the infinitesimal parameter $\epsilon \to 0$ in such way to conserve its scale invariance

$$W^{(\epsilon)}_{s\to(i,k)}(x,x_1,x_2) = 2\,\frac{x^{1-2\epsilon}}{(x_1\,x_2)^{1-\epsilon}}\,,\;\; W^{(\epsilon)} = \int_0^x\frac{x^{1-2\epsilon}\,d\,x_1}{x_1^{1-\epsilon}\,(x-x_1)^{1-\epsilon}} \approx \frac{2}{\epsilon}\,. \tag{240}$$

In this case the characteristic equation is more complicated

$$\frac{d}{d\xi}\,\phi^{(\epsilon)}(x,\xi) = -W^{(\epsilon)}\,\phi^{(\epsilon)}(x,\xi) + 2\int_0^x\frac{x^{1-2\epsilon}\,d\,x_1}{x_1^{1-\epsilon}\,(x-x_1)^{1-\epsilon}}\,\phi^{(\epsilon)}(x_1,\xi)\,\phi^{(\epsilon)}(x-x_1,\xi)\,. \tag{241}$$

We are not going to solve it. Note, that for the considered model the inclusive parton distribution can be easily calculated

$$D(x) = \int_{-i\infty}^{i\infty}\frac{d\,j}{2\pi i}\,\left(\frac{1}{x}\right)^j\,e^{w_j\,\xi}\,. \tag{242}$$

7.8 Deep-inelastic electron scattering off the polarized proton

One of the most interesting processes is the deep-inelastic electron scattering off the polarized target. We consider the scattering off the proton. The differential cross-section for finding an electron with a definite momentum in the final state is expressed in terms of the antisymmetric tensor describing the imaginary part of the forward scattering amplitude for the virtual photons with momenta q and polarization indices μ and ν

$$W^A_{\mu\nu} = \frac{1}{2}\,(W_{\mu\nu} - W_{\mu\nu}) = \frac{i}{pq}\,\epsilon_{\mu\nu\lambda\sigma}\,q^\lambda\,\left(s^\sigma\,g_1(x,Q^2) + s^\sigma_\perp\,g_2(x,Q^2)\right)\,, \tag{243}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the completely anti-symmetric tensor ($\epsilon_{0\,1\,2\,3} = 1$). The quantities $g_i(x,Q^2)$ ($i=1,2$) are the corresponding structure functions. The four-dimensional pseudo-vector s^σ describes the spin state of the completely polarized proton with the wave function $U(p)$

$$s^\sigma = \bar{U}(u)\gamma^\sigma\gamma_5 U(p) = -2m\,a^\sigma\,,\;\; \bar{U}(u)U(p) = 2m\,. \tag{244}$$

The vector a^σ is a parameter of the proton density matrix

$$U(p)\,\bar{U}(p) = \frac{1}{2}\,(\hat{p}+m)(1-\gamma_5\,\hat{a}),\;\gamma_5 = -i\,\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{245}$$

and has the properties

$$a\,p = 0\,,\;\; a^2 = -1\,,\;\; a_\perp = a - \frac{a\,q}{p\,q}\,p\,. \tag{246}$$

The tensor $W^A_{\mu\nu}$ is proportional to the imaginary part of the photon-proton scattering amplitude $T^A_{\mu\nu}$

$$W^A_{\mu\nu} = \frac{1}{\pi}\,\Im\,T^A_{\mu\nu} \tag{247}$$

having the similar representation

$$T^A_{\mu\nu} = \frac{1}{2}\,(T_{\mu\nu} - T_{\mu\nu}) = \frac{i}{pq}\,\epsilon_{\mu\nu\lambda\sigma}\,q^\lambda\,\left(s^\sigma\,\bar{g}_1(x,Q^2) + s^\sigma_\perp\,\bar{g}_2(x,Q^2)\right)\,. \tag{248}$$

With the use of the operator product expansion for the product of the electromagnetic currents

$$J_\mu(x) = \sum_q\,e_q\,\bar{\psi}_q(x)\,\gamma_\mu\,\psi_q(x) \tag{249}$$

entering in the expression for $T_{\mu\nu}$

$$T_{\mu\nu} = i\int d^4x\,e^{iqx}\;<p|T(J_\mu(x)\,J_\nu(0))|p> \tag{250}$$

in the Bjorken limit $q^2 \sim 2pq \ll m^2$ on the light cone $x^2 \sim q^{-2}$ we obtain [27]

$$\begin{aligned} T^A_{\mu\nu} = -i\,\epsilon_{\mu\nu\rho\sigma}\,\frac{q^\rho}{Q^2}\,\sum_{n=0}^{\infty}\,(1+(-1)^n)\,\left(\frac{2}{Q}\right)^n\,q_{\mu_1}...q_{\mu_n}\,\sum_k\varphi^k_{1,n}(Q^2)\;<h|R^k_{1\sigma\mu_1...\mu_n}|h> \\ +\frac{i}{Q^2}\,\left(\epsilon_{\mu\rho\lambda\sigma}\,q_\nu q^\rho - \epsilon_{\nu\rho\lambda\sigma}\,q_\mu\,q^\rho - q^2\,\epsilon_{\mu\nu\lambda\sigma}\right)\,\sum_{n=0}^{\infty}\,(1+(-1)^n) \\ \times q_{\mu_1}...q_{\mu_{n-1}}\,\sum_k\varphi^k_{2,n}(Q^2)\;<h|R^k_{2\sigma\lambda\mu_1...\mu_{n-1}}|h>\;. \end{aligned} \tag{251}$$

Here the index k enumerates various pseudo-tensor local operators R^k_i (i=1,2) with the positive charge parity. They are constructed from the quark and gluon fields in a gauge invariant way. The quantities $\varphi^k_{i,n}(Q^2)$ are corresponding coefficient functions. We should take into account such operators R^k_i which would lead to the largest asymptotic contribution to $T^A_{\mu\nu}$ in the Bjorken limit. In particular, it means, that the tensors $R^k_{1\sigma\mu_1...\mu_n}$ and $R^k_{2\sigma\lambda\mu_1...\mu_{n-1}}$ should be symmetric and traceless in indices σ, μ_1, ..., μ_n and $\lambda\mu_1...\mu_{n-1}$, respectively. Further, they should have the minimal dimensions d in the units of mass for a given number of indices. Examples of such operators are given below

$$R^q_{1\sigma\mu_1...\mu_n} = i^n\,S_{\{\sigma\,\mu_1...\mu_n\}}\,\bar{\psi}\gamma_5\,\gamma_\sigma\,D_{\mu_1}...D_{\mu_n}\,\psi_q\,, \tag{252}$$

$$R^q_{2\sigma\lambda\mu_1...\mu_{n-1}} = i^n\,S_{\{\mu_1...\mu_{n-1}\}}\,A_{[\sigma\,\mu_1]}\,\bar{\psi}\gamma_5\,\gamma_\sigma\,D_{\mu_1}...D_{\mu_{n-1}}\,\psi_q\,, \tag{253}$$

where the sign S means the symmetrization of the tensor in the corresponding indices and subtraction of traces. The sign A implies its anti-symmetrization. In the above expression D_μ is the covariant derivative

$$D_\mu = \partial_\mu + ig\,A_\mu\,, \tag{254}$$

where $A = t_a\,A^a$ and t_a are the color group generators in the fundamental representation. In principle each covariant derivative can be applied to the left quark field with an opposite sign

without changing the value of the matrix element between the initial and final proton states having the same momentum p. For the deep-inelastic ep scattering only the operators with even values of n are taken into account because they have the positive charge parity. The tensor $R^q_{1\sigma\mu_1\ldots\mu_n}$ symmetrized over all indices has the Lorentz spin $j = n+1$ and the canonical dimension $d = 3 + n$. Further, due to one anti-symmetrization the tensor $R^q_{2\sigma\lambda\mu_1\ldots\mu_{n-1}}$ has the Lorentz spin $j = n$ and the canonical dimension $d = 3 + n$. The difference between the canonical dimension and the Lorentz spin of an operator is its twist

$$t = d - j\,. \tag{255}$$

It is obvious, that the twists of operators R^q_1 and R^q_2 are $t = 2$ and $t = 3$, respectively. Increasing the value of the twist leads usually to diminishing the power of Q for the corresponding contribution in the cross-section. However for the deep inelastic scattering off the polarized target the above two operators give a comparable contribution to the structure function $g_2(x, Q^2)$.

The matrix elements of the operators $R^{q'}_i$ between the free quark states in the Born approximation can be easily calculated

$$< q|R^{q'}_{1\sigma\mu_1\ldots\mu_n}|q >= -\delta_{q\,q'}\, S_{\{\mu_1\ldots\mu_n\}} \left(\frac{s_\sigma}{n+1}\, p_{\mu_1}\ldots p_{\mu_n} + \frac{n}{n+1}\, s_{\mu_1}\, p_\sigma\, p_{\mu_2}\ldots p_{\mu_n}\right) \tag{256}$$

and

$$< q|R^{q'}_{2\sigma\mu_1\ldots\mu_n}|q >= -\frac{1}{2}\,\delta_{q\,q'}\, S_{\{\mu_1\ldots\mu_n\}}\left(s_\sigma\, p_{\mu_1}\ldots p_{\mu_n} - s_{\mu_1}\, p_\sigma\, p_{\mu_2}\ldots p_{\mu_n}\right)\,. \tag{257}$$

In the Born approximation the structure functions are

$$g_1(x) = \frac{1}{2}\, e_q^2\, \delta(x-1)\,, \quad g_2(x) = 0\,,$$

$$\bar{g}_1(x) = e_q^2\, \frac{x}{x^2 - 1 - i\,0}\,, \quad \bar{g}_2(x) = 0\,, \tag{258}$$

The formulas of the operator product expansion give the same result for $\bar{g}_i(x)$ providing that the coefficient functions in the same approximation are

$$\varphi^q_{1,n} = e_q^2\,, \;\; \varphi^q_{2,n} = 2\left(\frac{n}{n+1}\right)^2 e_q^2\,. \tag{259}$$

We used here the identity

$$q_\nu q_\rho \epsilon_{\mu\rho\lambda\sigma} - q_\mu q_\rho \epsilon_{\nu\rho\lambda\sigma} - q^2 \epsilon_{\mu\nu\lambda\sigma} = q_\rho\, \left(q_\sigma \epsilon_{\mu\nu\rho\lambda} - q_\lambda \epsilon_{\mu\nu\rho\sigma}\right)\,. \tag{260}$$

Because the form of the operator product expansion does not depend on the target, one can construct the photon-hadron scattering amplitude $T^A_{\mu\nu}$ with the use of the same coefficient functions $\varphi^q_{i,n}$ as follows

$$T^A_{\mu\nu} = -i\,\epsilon_{\mu\nu\rho\sigma}\,\frac{q^\rho}{Q^2}\sum_{n=0}^{\infty}\sum_q e_q^2\,(1 + (-1)^n)\left(\frac{1}{x}\right)^n < h\left|R^q_{1\sigma\ldots} + \frac{2\,n}{n+1}\,R^q_{2\sigma\ldots}\right|h >\,. \tag{261}$$

Here we introduced the light-cone projections of the Lorenz tensors defined below

$$O_{\ldots} = \frac{q'^{\mu_1}}{pq}\ldots\frac{q'^{\mu_n}}{pq}\,O_{\mu_1\ldots\mu_n}\,,\;\; q' = q - xp\,,\;\; q'^2 = 0 \tag{262}$$

and used the relation

$$\frac{n}{n+1}\,\frac{q_\rho}{pq}\,(q_\sigma\epsilon_{\mu\nu\rho\lambda}-q_\lambda\epsilon_{\mu\nu\rho\sigma})\,\frac{q^{\mu_1}}{pq}...\frac{q^{\mu_{n-1}}}{pq}\,R^q_{2\sigma\lambda\mu_1...\mu_{n-1}}=-\epsilon_{\mu\nu\rho\sigma}q^\rho\,R^q_{2\sigma...} \tag{263}$$

valid for matrix elements between the states with the same momentum p.

Note, that for the case, when the current quark mass is large, there is another twist-3 operator

$$R^q_{3\,\sigma\mu_1...\mu_n}=m_q\,i^{n-1}\,S_{\{\mu_1...\mu_n\}}\,A_{\sigma\mu_1}\,\bar\psi\gamma_5\,\gamma_\sigma\,\gamma_{\mu_1}D_{\mu_2}...D_{\mu_{n-1}}\,\psi_q\,. \tag{264}$$

Moreover, in the Born approximation one can obtain for the matrix elements of these twist-3 operators between the quark states the relation

$$<q|R^{q'}_{3\,\sigma\mu_1...\mu_n}|q>=2\;<q|R^{q'}_{3\,\sigma\mu_1...\mu_n}|q>\;. \tag{265}$$

However the above expression for the amplitude $T^A_{\mu\nu}$ written in terms of the Wilson operator expansion remains valid also in the case of massive quarks. To verify it one should calculate also its matrix elements between the states containing the gluons.

For this purpose with the use of the Heisenberg equation of motion for the quark fields

$$(i\hat D-m_q)\,\psi_q=0\,,\;\;\bar\psi(i\hat D-m_q)=0 \tag{266}$$

we can obtain the following operator identity

$$R^\perp_{2\,\sigma...}-\frac{1}{2}\,R^\perp_{3\,\sigma...}=\frac{1}{2n}\,\sum_{l=1}^{n-1}(n-l)\,Y^{\perp(l)}_{\sigma...}\,, \tag{267}$$

where the operators are constructed from the fields $\bar\psi,\psi,G_{\rho\lambda}$ and their covariant derivatives. For the deep-inelastic scattering, where only operators with the positive charge parity are essential, n is even and in fact only the following linear combinations of $Y^{\perp(l)}_{\sigma...}$ appear

$$\begin{aligned}&Y^{\perp\,(n-l)}_{\sigma...}-Y^{\perp\,(l)}_{\sigma...}=R^{\perp\,(l)}_{4\,\sigma...}-R^{\perp\,(n-l)}_{4\,\sigma...}\,,\\&R^{\perp\,(l)}_{4\,\sigma...}=\bar\psi\gamma_5\gamma_.(iD_.)^l g\,G^\perp_{\sigma.}\,(iD_.)^{n-l-1}\psi\end{aligned} \tag{268}$$

and

$$\begin{aligned}&Y^{\perp\,(n-l)}_{\sigma...}+Y^{\perp\,(l)}_{\sigma...}=R^{\perp\,(l)}_{5\,\sigma...}+R^{\perp\,(n-l)}_{5\,\sigma...}\,,\\&R^{\perp\,(l)}_{5\,\sigma...}=\bar\psi\gamma_.(iD_.)^l g\,\widetilde G^\perp_{\sigma.}\,(iD_.)^{n-l-1}\psi\,,\end{aligned} \tag{269}$$

where

$$G_{\rho\lambda}=t_aG^a_{\rho,\lambda}\,,\;\;\widetilde G_{\rho\lambda}=\frac{1}{2}\,\epsilon_{\rho\lambda\sigma\eta}\,G^{\sigma\eta}\,,\;\;G^a_{\rho,\lambda}=\partial_\rho A^a_\lambda-\partial_\lambda A^a_\rho+igf^{abc}A^c_\rho A^b_\lambda\,. \tag{270}$$

By calculating the matrix elements of the operators $R_{2\,\sigma...}$ between the quark and quark-gluon states with the use of the operator identity (267) one can verify, that indeed the operator product expansion in the Born approximation is given by the above expression. We confirm this result below also with the use of the parton ideas.

To begin with, we remind, that the numerator of the gluon propagator $D_{\mu\nu}(k)$ in the axial gauge $q'_\mu A^\mu=0$ and the numerator of the quark propagator $G(k)$ for positive energies $kq'/pq'=\beta>0$ can be expressed in terms of projectors on physical states with two helicities $\lambda=\pm s$

$$-\delta_{\mu\nu}+\frac{k_\mu q'_\nu+k_\nu q'_\mu}{kq'}=\sum_{\lambda=\pm1}e^\lambda_\mu(k')\,e^{\lambda\,*}_\nu(k')+k^2\,\frac{q'_\mu q'_\nu}{(kq')^2}\,, \tag{271}$$

$$\hat{k} + m_q = \sum_{\lambda=\pm 1/2} u_\lambda(k')\bar{u}_\lambda(k') + \frac{k^2 - m_q^2}{s\beta}\,\hat{q}' \tag{272}$$

providing that the last contributions proportional to the particle virtuality are small. In the above relations we have for the gluon and quark momenta

$$k' = k - \frac{k^2}{2kq'}\,q' = \beta p + k_\perp - \frac{k_\perp^2}{\beta s}\,q'\,,\ k'^2 = 0 \tag{273}$$

and

$$k' = k - \frac{k^2 - m_q^2}{2kq'}\,q' = \beta p + k_\perp - \frac{k_\perp^2 - m_q^2}{\beta s}\,q'\,,\ k'^2 = m_q^2\,, \tag{274}$$

respectively.

For the longitudinally polarized hadron, where

$$a_L = \frac{1}{m}\,\lambda_h\,\left(p - \frac{m^2}{pq'}\,q'\right)\,,\ a_L^2 = -1\,,\ \lambda_= \pm 1\,, \tag{275}$$

the virtual gluon propagators in the light-cone gauge are not attached to the quark line situated between the photon vertices γ_μ and γ_ν, because such contribution would lead to an additional large denominator in the quark propagators, which is not compensated by a nominator. As for the quark lines, the numerators in their propagators can be simplified as follows

$$\begin{aligned} \gamma_\mu(\hat{k} + \hat{q} + m)\gamma_\nu &\to -i\epsilon_{\mu\nu\lambda\sigma}q^\lambda\gamma_5\gamma^\sigma \\ \gamma_\nu(\hat{k} - \hat{q} + m)\gamma_\mu &\to -i\epsilon_{\mu\nu\lambda\sigma}q^\lambda\gamma_5\gamma^\sigma\,. \end{aligned} \tag{276}$$

It leads to the following expression for the cross-section

$$W^A_{\mu\nu} = -\frac{i}{pq}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda \sum_q \frac{1}{2}\,e_q^2 \int < \gamma_5\gamma^\sigma > \,(\delta(\beta - x) + \delta(\beta + x))\,d\beta\,, \tag{277}$$

where $< \gamma_5\gamma^\sigma >$ implies the corresponding vertex, in which the integration over the Sudakov parameter β for the neighboring virtual quark is not performed. The second term in the bracket describes the anti-quark contribution in the Dirac picture, where $\beta < 0$. Of course, the physical anti-quark energy is positive. Because for the longitudinal polarization s_h the indices μ, ν are transversal, γ^σ is effectively multiplied by q'^σ, which allows us to leave in the neighboring quark propagators only projectors on the physical states with two helicities leading to the result

$$\frac{1}{s}\int < \gamma_5\hat{q}' > (\delta(\beta - x) + \delta(\beta + x))\,d\beta = \sum_{r=q,\bar{q}} \left(n_r^+(x) - n_r^-(x)\right)\,. \tag{278}$$

Thus, we obtain the parton expression for the structure function $g_1(x)$ in terms of quark and anti-quark distributions with helicities $\lambda = \pm 1/2$

$$g_1(x) = \sum_{r=q,\bar{q}} \frac{e_r^2}{2}\,\lambda_h\,\left(n_r^+(x) - n_r^-(x)\right)\,. \tag{279}$$

Let us consider now a transversely polarized hadron. Taking into account, that in this case one of two indices μ and ν is transversal and other is longitudinal, corresponding to

the substitution of one of two photon vertices by $\hat{q}'$, we can calculate the intermediate quark propagator obtaining the relation

$$(g_1(x)+g_2(x))\,S^{\perp}_{\sigma} = -\sum_q \frac{1}{2}\,e_q^2 \int d\,\beta \;<\gamma_5\gamma^{\perp}_{\sigma}>\;(\delta(\beta-x)+\delta(\beta+x))\;. \tag{280}$$

It coincides in fact with the result, which can be derived from the expression for $T_{\mu\nu}$ written in terms of the operator product expansion. To show it in the light-cone gauge $A_{\cdot}=0$ we apply the operator identity

$$A^q_{\sigma\ldots} \equiv R^q_{1\,\sigma\ldots} + \frac{2n}{n+1}\,R^q_{2\,\sigma\ldots} = \bar{\psi}_q\gamma_5\gamma_\sigma(i\partial_{\cdot})^n\psi_q\;. \tag{281}$$

Further, using the relation

$$<h|A^q_{\sigma\ldots}|h> = \int d\beta\,\beta^n\;<\gamma_5\gamma_\sigma> \tag{282}$$

one can calculate $W^A_{\mu\nu}$

$$W^A_{\mu\nu} = -\frac{i}{pq}\,\epsilon_{\mu\nu\lambda\sigma}\,q^\lambda\sum_q\frac{1}{2}\,e_q^2\,\frac{1}{\pi}\Im\sum_{n=0}^{\infty}(1+(-1)^n)\left(\frac{1}{x}\right)^n <h|A^q_{\sigma\ldots}|h>$$

$$= -\frac{i}{pq}\,\epsilon_{\mu\nu\lambda\sigma}q^\lambda\sum_q\frac{1}{2}\,e_q^2\int <\gamma_5\gamma^\sigma>\;(\delta(\beta-x)+\delta(\beta+x))\;d\,\beta\;. \tag{283}$$

Note, that the operator $A^q_{\sigma\ldots}$ does not have a definite twist and therefore it is not renormalized in a multiplicative way. Therefore it is more consistent to use the representation of $W_{\mu\nu}$ in terms of the operators $R_{1\,\sigma\ldots}$ and $R_{2\,\sigma\ldots}$ Due to the relativistic invariance the different components of the vector $R_{1\,\sigma\ldots}$ are proportional. For example,

$$<h|R^{\perp}_{1\,\sigma\ldots}|h> = s^{\perp}_{\sigma}\,\frac{pq'}{sq'}\,\frac{1}{n+1}\;<h|R_{1\,\ldots}|h>\;. \tag{284}$$

Analogously for the operator $A^{q\,\perp}_{\sigma\ldots}$ with the use of equations of motion we obtain

$$<h|A^{q\,\perp}_{\sigma\ldots}|h> = s^{\perp}_{\sigma}\,\frac{pq'}{sq'}\,\frac{1}{n+1}\;<h|R_{1\,\ldots}|h>$$

$$+\frac{n}{n+1}\;<h|R^{\perp}_{3\,\sigma\ldots}|h> + \frac{1}{n+1}\sum_{l=1}^{n-1}(n-l)\;<h|Y^{l\,\perp}_{\sigma\,\ldots}|h>\;. \tag{285}$$

One can introduce the generating functions $E(\beta)$, $A(\beta)$, $C(\beta)$ and $Y(\beta_1,\,\beta_2)$ describing the matrix elements of the corresponding operators

$$\lambda_h\int d\beta\beta^n\;<\frac{\gamma_5\bar{q}'}{2pq'}> = \int d\beta\beta^n\,E(\beta)\,,$$

$$<h|A^{\perp}_{\sigma\,n}|h> = \int d\beta\,\beta^n\;<\gamma_5\,\gamma^{\perp}_{\sigma}> \equiv s^{\perp}_{\sigma}\int d\beta\,\beta^{n-1}\,A(\beta)\,,$$

$$<h|R^{\perp}_{3\sigma\,\ldots}|h> = \int d\beta\,\beta^{n-1}\;<\gamma_5\,\gamma^{\perp}_{\sigma}\gamma_{\cdot}> \equiv s^{\perp}_{\sigma}\int d\beta\,\beta^{n-1}\,C(\beta)\,,$$

$$< h|Y^{l\,\perp}_{\sigma\,\ldots}|h >\equiv \int d\beta_1 d\beta_2\, \beta_1^{l-1}\, \beta_2^{n-l-1}\, Y(\beta_1,\, \beta_2)\,. \tag{286}$$

We have in particular

$$g_1(x) = \sum_q \frac{e_q^2}{2x}\, (E_q(x) - E_q(-x))\ ,$$

$$g_1(x) + g_2(x) = -\sum_q \frac{e_q^2}{2x}\, (A_q(x) - A_q(-x))\ . \tag{287}$$

Further, due to the equation of motion one obtains the relation among these functions

$$\left(1 - \beta\,\frac{d}{d\beta}\right)\, A(\beta) = E(\beta) - \beta\frac{d}{d\beta}\, C(\beta) + \beta \int \frac{d\beta_1}{\beta_1 - \beta}\left(\frac{\partial}{\partial\beta}\, Y(\beta_1,\, \beta) + \frac{\partial}{\partial\beta_1}\, Y(\beta,\, \beta_1)\right)\ . \tag{288}$$

For the solution of this differential equation concerning $A(\beta)$ the integration constant should be chosen from the Cottingem sum rule

$$\int_0^1 g_2(x)dx = 0 \tag{289}$$

equivalent to the equality

$$\int_{-1}^1 \frac{d\beta}{\beta}\, A(\beta) = \int_{-1}^1 \frac{d\beta}{\beta}\, A(\beta)\,. \tag{290}$$

Thus, we obtain the relation

$$g_1(x) + g_2(x) = \int_x^1 \frac{dx'}{x'}\, g_1(x') - \sum_q \frac{e_q^2}{2} \int_x^1 \frac{dx'}{x'}\, K(x')\,,$$

$$K(x') = -\frac{d}{dx'}\, C_q(x') + \int \frac{dx_1}{x_1 - x'}\left(\frac{\partial}{\partial x'} Y_q(x_1, x') + \frac{\partial}{\partial x_1} Y_q(x', x_1)\right) - (x' \to -x')\,. \tag{291}$$

Providing that the current quark masses are small and the gluon contributions is not large, one can neglect the term containing $K(x')$ in the above expression. In this case it is reduced to the Wilczek-Mandula relation

$$g_1(x) + g_2(x) \approx \int_x^1 \frac{dx'}{x'}\, g_1(x')\,, \tag{292}$$

which appears if we would take into account only the contribution of the twist-2 operator.

Note, that the matrix element of the current $\gamma_5\gamma_\sigma^\perp$ between hadron states contains also the contribution from the off-mass shell quarks, because we can not neglect the additional term $\sim \hat{q}'$ in their propagators. To solve this problem let us use the following identity for this current

$$\gamma_5\gamma_\sigma^\perp = \gamma_5 \frac{\hat{q}' k_\sigma^\perp}{kq'} + m_q\, \gamma_5\gamma_\sigma^\perp \frac{\hat{q}'}{kq'} - \frac{(\hat{k} - m_q)\, \gamma_5 \hat{q}' \gamma_\sigma^\perp}{2kq'} - \frac{\gamma_5 \hat{q}' \gamma_\sigma^\perp (\hat{k} - m_q)}{2kq'}\,. \tag{293}$$

In the last two terms the factors $(\hat{k} - m_q)$ cancel the nearest quark propagator and the corresponding gluon-quark vertex $g\, t^a\, \gamma_\rho$ turns out to be at the same space-time point as the current producing the new vertices

$$D_1(\beta_1, \beta)\, s_\sigma^\perp =< g\, \hat{A}^\perp\, \frac{\gamma_5 \bar{q}' \gamma_\sigma^\perp}{pq'} >,\ \ D_2(\beta, \beta_1)\, s_\sigma^\perp =< g\, t^a\, \frac{\gamma_5 \bar{q}' \gamma_\sigma^\perp}{pq'} \hat{A}^\perp >\,, \tag{294}$$

in which the integration over the Sudakov parameters β and β_1 respectively for ingoing and outgoing quarks is not performed and $A^{\perp}_{\rho}$ means the transverse component of the gluon field in the momentum space. Such procedure corresponds to the use of the equation of motion in the perturbation theory and leads to the relation

$$A(\beta) = B(\beta) + C(\beta) - \int d\beta_1 D(\beta_1, \beta)\,, \tag{295}$$

where

$$D(\beta_1, \beta) = \frac{1}{2}\,\left(D_1(\beta_1,\,\beta) + D_2(\beta,\,\beta_1)\right)\,. \tag{296}$$

Here we introduced the functions $B(\beta)$ and $C(\beta)$ according to the following definitions

$$B(\beta)\, s^{\perp}_{\sigma} =< \gamma_5\, \frac{\hat{q}'\, k_{\sigma}}{pq'} >\,,\;\; C(\beta)\, s^{\perp}_{\sigma} =< m_q\, \gamma_5\, \gamma^{\perp}_{\sigma} \frac{\hat{q}'}{pq'} >\;. \tag{297}$$

It is important, that in the vertices B, C, D_1, D_2 the nearest quarks can be considered as real particles with momenta k' and two physical helicities, because the extra term $\sim \hat{q}'$ in numerators of their propagators does not give any contribution due to the relation $\hat{q}'\hat{q}' = 0$. Similarly the gluon absorbed in the vertices D_1, D_2 in the axial gauge also can be considered as a real particle with two possible polarizations, because the extra term $\sim q'_{\mu} q'_{\nu}$ in the numerator of its propagator gives the vanishing contribution. The corresponding operators are simple examples of a large class of quasi-partonic operators of arbitrary twists which will be considered below.

7.9 Evolution equations for quasi-partonic operators

As it was demonstrated in the previous subsection, with the use of the equations of motion one can reduce the set of all twist-3 operators appearing in the operator product expansion of two electromagnetic currents in the light cone gauge

$$A_{\mu} q'^{\mu} = A_{.} = 0\,,\;\; q' = q + xp \tag{298}$$

to the class of operators whose matrix elements can be calculated between on-mass shell parton states (quarks and gluons) having two helicities. It turns out, that almost all twist-4 operators appearing in the power corrections $\sim 1/Q^2$ to structure functions of the deep-inelastic lepton-hadron scattering also can be reduced to this class of quasi-partonic operators (QPO). Apart from the on-mass shell requirement it is natural also to impose the additional constraint on QPO: they should not contain explicitly the strong coupling constant [30]. Indeed, in the parton model the form of strong interactions is essential only when one calculates parton distributions whereas the matrix elements of the operators between parton states do not depend on the QCD dynamics. Below we shall neglect the quark mass. It will give us a possibility to use the conformal symmetry of the theory.

The mass-shell condition will be fulfilled in the light-cone gauge providing that corresponding QPO are invariant under the following field transformations

$$\psi \to \psi + \gamma_{.}\chi\,,\;\; \bar{\psi} \to \bar{\psi} + \bar{\chi}\gamma_{.}\,,\;\; A_{\mu} \to A_{\mu} + q'_{\mu}\,\phi\,, \tag{299}$$

where $\chi, \bar{\chi}$ and ϕ are arbitrary spinor and scalar functions. Therefore these quantities can be constructed as a product of the structures (we also take into account the independence from the QCD coupling constant g)

$$\gamma_{.}\psi\,,\;\; \bar{\psi}\gamma_{.}\,,\;\; G^{\perp}_{.\mu} = \partial_{.} A^{\perp}_{\mu}\,,\;\; D_{.} = \partial_{.}\,,\;\; \gamma_{.}\,. \tag{300}$$

Of course, the composite operators should be colorless, but on an intermediate step we consider the operators with an open color as it takes place for above structures. In principle the fields $\psi, \bar{\psi}$ can belong to an arbitrary representation of the color group. For example, in the super-symmetric models gluinos are transformed according to the adjoint representation of the gauge group. But as a rule the fields ψ and $\bar{\psi}$ are assumed to be usual massless quark fields.

Below we list examples of QPO for the twist-2

$$\bar{\psi}\gamma_.(i\partial_.)^n\psi\,,\ \bar{\psi}\gamma_5\gamma_.(i\partial_.)^n\psi\,,\ \bar{\psi}\gamma_.\gamma_\rho^\perp(i\partial_.)^n\psi\,,$$

$$\left(-i\partial_. A_\rho^\perp\right)(i\partial_.)^{n+1}A_\rho^\perp\,,\ \epsilon_{\rho_1\rho_2}^{\perp\perp}\left(-i\partial_. A_{\rho_1}^\perp\right)(i\partial_.)^{n+1}A_{\rho_2}^\perp\,,\ S_{\rho_1\rho_2}\left(-i\partial_. A_{\rho_1}^\perp\right)(i\partial_.)^{n+1}A_{\rho_2}^\perp\,, \tag{301}$$

twist-3

$$\left((-i\partial_.)^{n_1}\bar{\psi}\right)\gamma_.\gamma_5(i\partial_. A_\sigma^\perp)(i\partial_.)^{n_2}\psi\,,\ f_{abc}\epsilon_{\rho_1\rho_2}^{\perp\perp}\left((-i\partial_.)^{n_1}A_{\rho_1}^{a\perp}\right)(i\partial_. A_\sigma^{b\perp})(i\partial_.)^{n_2}A_{\rho_2}^{c\perp}\,, \tag{302}$$

and twist-4

$$\left((-i\partial_.)^{n_1}\bar{\psi}\right)\gamma_.\left((i\partial_.)^{n_2}\psi\right)\left((-i\partial_.)^{n_3}\bar{\psi}\right)\gamma_.(i\partial_.)^{n_4}\psi\,,$$

$$\left((-i\partial_.)^{n_1}A_\rho^\perp\right)\left((i\partial_.)^{n_2}A_\rho^\perp\right)\left((-i\partial_.)^{n_3}A_\sigma^\perp\right)(i\partial_.)^{n_4}A_\sigma^\perp\,, \tag{303}$$

where $\epsilon_{\rho_1\rho_2}^{\perp\perp}$ is an anti-symmetric tensor in the two-dimensional space ($\epsilon_{12}^{\perp\perp}=1$).

One can easily verify, that the twist of QPO coincides with the number k of constituent fields ψ, $\bar{\psi}$, $A_\sigma^\perp$

$$t = d - j = k\,. \tag{304}$$

They can be written in the form

$$O^{\{r\}} = \Gamma_{\rho_1\ldots\rho_l}^{\{r\}}\prod_{t=1}^{l}(i\partial_.)^{n_t+s_t-1}\,\varphi_{\rho_t}^{r_t}\,, \tag{305}$$

where $\varphi_{\rho_t}^{r_t}$ is one of these fields with the spin index ρ_t and the color index r_t. The factor $\Gamma^{\{r\}}$ is a numerical matrix satisfying the constraints

$$q'_{\rho_t}\,\Gamma_{\rho_1\ldots\rho_l}^{\{r\}} = 0\,,\ \hat{q}'_{\bar{\rho}\rho_t}\,\Gamma_{\rho_1\ldots\rho_l}^{\{r\}} = 0 \tag{306}$$

in vector an spinor indices. Generally we consider the matrix elements of such operators between the hadron states h, h' with the different momenta p_h, $p_{h'}$ [30]

$$< h'|O|h >\,, \tag{307}$$

They appear, for example, in the theoretical description of the electro- and photo- production of photons [34]. The simple case of the matrix element between the vacuum and the hadron state

$$< 0|O|h > \tag{308}$$

corresponds to the calculation of the hadron wave functions at small parton distances $x^2 \sim 1/Q^2 \to 0$ [23, 24, 25] appearing in the theory of hadron form-factors and other exclusive processes at large momentum transfers.

To have the universal formulas for particles and anti-particles it is convenient to change the normalization of the quark spinors and gluon polarization vectors by introducing the new wave functions [30]

$$\xi^\lambda(k) = \frac{\hat{k}\hat{q}'}{s}\, u^\lambda(p')\,,\ s = 2p'q'\,,\ k^2 = s\alpha\beta + k_\perp^2 = 0\,,$$

$$\eta_\rho^\lambda(k) = \beta\, e_\rho^\lambda(p') - \frac{2}{s}\, q'_\rho\, k_\perp^\sigma e_\sigma^\lambda\,,\ \lambda = \pm 1\,,\ p = p' + \frac{m_h^2}{s}\, q'\,, \tag{309}$$

where the basis spinors $u^\lambda(p')$ and vectors $e^\lambda(p')$ satisfy the following eigenvalue equations

$$\gamma_5 u^\lambda(p') = -\lambda u^\lambda(p')\,,\ \hat{p}'\, u^\lambda(p') = 0\,,\ \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\,,\ p'_0 > 0\,,$$

$$i\epsilon_{\rho\tau}^{\perp\perp}\, e_\tau^\lambda(p') = \lambda e_\tau^\lambda(p')\,,\ p' e_\tau^\lambda(p') = q' e_\tau^\lambda(p') = 0\,,\ \epsilon_{\rho\tau}^{\perp\perp} = \frac{2}{s}\, \epsilon_{\rho\tau\alpha\beta} p'_\alpha q'_\beta \tag{310}$$

and the normalization conditions

$$\bar{\xi}^{\lambda'}(k)\gamma_\mu \xi^\lambda(k) = 2\beta k_\mu \delta_{\lambda\lambda'}\,,\ \eta_\rho^{\lambda'}(k)\, \eta_\rho^\lambda(k) = -\beta^2\, \delta_{\lambda\lambda'}\,. \tag{311}$$

Inverse transformations for spinors with positive and negative energies and for the polarization vectors is

$$u^\lambda(k) = \beta^{-1/2}\, \xi^\lambda(k)\,,\ v^\lambda(-k) = \beta^{-1/2}\, \gamma_2 \xi^{-\lambda *}(k)\,,\ e^\lambda(k) = \beta^{-1} \eta^\lambda(k)\,. \tag{312}$$

It is helpful also to impose on $u^\lambda(p')$ and $e^\lambda(p')$ additional constraints by fixing their phases to satisfy the equalities

$$v^\lambda(-k) = |\beta|^{-1/2}\, \xi^{-\lambda}(-k)\,,\ \eta^{\lambda *}(k) = \eta^{-\lambda}(-k)\,. \tag{313}$$

The above relations allow us to relate analytically the amplitude for the particle production to that for the annihilation of the anti-particle with an opposite helicity. In such normalization the numerators of the propagators for the partons entering in the vertex can be written as follows

$$\hat{k} = \beta^{-1} \sum_{\lambda=\pm 1} \xi^\lambda(k') \bar{\xi}^\lambda(k') + \frac{k^2}{s\beta}\, \hat{q}'\,,\ k' = k - \frac{k^2}{\beta s}\, q'\,,$$

$$-\delta_{\mu\nu} + \frac{k_\mu q'_\nu + k_\nu q'_\mu}{kq'} = \beta^{-2} \sum_{\lambda=\pm 1} \eta_\mu^\lambda(k') \eta_\nu^{\lambda *}(k') + \frac{4k^2}{s^2\beta^2}\, q'_\mu q'_\nu\,. \tag{314}$$

Because for QPO the last terms in the right hand sides of the above equalities give vanishing contributions one can express the hadron matrix elements of these operators as the sum of products of the matrix elements $O_{\lambda_1 \ldots \lambda_l}$ between the on-mass shell parton states with definite helicities $\lambda_1 \ldots \lambda_l$ integrated with the inclusive parton correlation functions $N_{\lambda_1 \ldots \lambda_l}(\beta_1, \ldots \beta_l)$

$$< h' | O^{\{r\}} | h > = \sum_{\lambda_1 \ldots \lambda_l} \int d\beta_1 \ldots d\beta_l\, O^{\{r\}}_{\lambda_1 \ldots \lambda_l} \prod_{t=1}^{l} \beta_t^{n_t}\, N^{\{r\}}_{\lambda_1 \ldots \lambda_l}(\beta_1, \ldots \beta_l)\,, \tag{315}$$

where

$$O^{\{r\}}_{\lambda_1 \ldots \lambda_l} = \Gamma^{\{r\}}_{\rho_1 \ldots \rho_l} \prod_{t=1}^{l} \beta_t^{-1} \xi_{\rho_t}^{\lambda_t}(k'_t)\,, \tag{316}$$

$$N^{\{r\}}_{\lambda_1...\lambda_l}(\beta_1,...\beta_l) = \int d^4k_1...d^4k_l M^{\{r\}}_{\rho_1...\rho_l}(k_1,...,k_l) \prod_{t=1}^{l} \frac{\bar{\xi}^{\lambda_t}_{\rho_t}(k'_t)}{k_t^2} \delta(\beta_t - \frac{k_t q'}{p'q'}) \,. \tag{317}$$

In the above expressions for simplicity we denote by $\xi^{\lambda_t}_{\rho_t}(k'_t)$ the parton wave functions which for various t could be $\xi^{\lambda}(k')$, $\bar{\xi}^{-\lambda}(-k')$ or $\eta^{\lambda}_{\rho}(k')$. The product $O^{\{r\}}_{\lambda_1...\lambda_l} \prod_{t=1}^{l} \beta_t^{n_t}$ corresponds to the parton matrix element of the operator $O^{\{r\}}_{...}$ which includes the derivatives $(i\partial_.)^{n_t}$ acting on the quark or gluon fields.

The parton correlation functions (PCF) $N^{\{r\}}_{\lambda_1...\lambda_l}(\beta_1,...\beta_l)$ contain the δ-function, corresponding to the energy-momentum conservation

$$\beta = \sum_{t=1}^{l} \beta_t \,, \tag{318}$$

where β is the difference of the Sudakov variables for ingoing and outgoing hadrons. In the case of the usual deep-inelastic scattering β is zero, but it is not zero for the case when momenta of initial and final hadrons are different. PCF can be calculated in terms of scalar products of parton wave functions for initial and final hadrons integrated over the momenta and summed over quantum numbers of unobserved partons. Generally the function $N^{\{r\}}_{\lambda_1...\lambda_l}(\beta_1,...\beta_l)$ describes several products of different wave functions in a correspondence with two possible signs for each β_t. Namely, for a positive sign the corresponding parton belongs to the initial state and for an opposite sign it is in the final state. Note, that PCF are similar to the density matrix in the quantum mechanics.

Another important property of QPO is that they are closed under the renormalization in the leading logarithmic approximation [30]. Namely, if one calculates the one-loop correction to their matrix elements between parton states, the resulting operators after the use of the equations of motion can be reduced again to QPO. Moreover, the integral kernel in the evolution equation for matrix elements of QPO is expressed as a sum of pair splitting kernels $\Phi^{r_i r_k}_{r_{i'} r_{k'}}$ [30]

$$\frac{\partial}{\partial \tilde{\xi}} N^{\{r\}}(\beta_1,...\beta_l) =$$

$$\sum_{i<k} \sum_{r_{i'} r_{k'}} \int d\beta_{i'}\, d\beta_{k'}\, \Phi^{r_i r_k}_{r_{i'} r_{k'}}(\beta_i,\beta_k|\beta_{i'},\beta_{k'})\, \delta\left(\beta_{i'} + \beta_{k'} - \beta_i - \beta_k\right)\, N^{\{r'\}}(\beta_1,...,\beta_{i'}...\beta_{k'}...\beta_l) \,. \tag{319}$$

Here indices r include the types of particles (quark or gluon) and their color, flavor and helicity. In an accordance with Ref.[30] we changed here the normalization of the "time" variable ξ in comparison with expression (79)

$$\tilde{\xi} = \frac{1}{b} \ln\left(1 + b\,\frac{g^2}{16\pi^2} \ln\frac{Q^2}{\Lambda^2_{QCD}}\right) \,,\; b = \frac{11}{3} N_c - \frac{2}{3} n_f \,. \tag{320}$$

All pair splitting kernels $\Phi^{r_i r_k}_{r_{i'} r_{k'}}$ are calculated for both signs of the Sudakov variables β_r [30]. They describe interactions of quarks and gluons in all possible helicity, colour and flavour states and do not contain infrared divergencies. The pair kernels with quantum numbers of quark or gluon have contributions from corresponding virtual particles in the t-channel. The evolution equations for PCF in the coordinate representation have many interesting properties including their conformal invariance under the Möbius transformations

$$z \to \frac{az+b}{cz+d} \,, \tag{321}$$

where $z = x^\mu q'_\mu$ is the light cone variable. This symmetry gives a possibility to find the eigenfunctions of the pair splitting kernels in terms of the Gegenbauer polynomials [32]

$$R^{\bar{q}q}_{n+1}(\beta_1,\beta_2) = \sum_{k=0}^{n} (-1)^k \, \frac{n!\,(n+2)!\,\beta_1^k\,\beta_2^{n-k}}{k!\,(k+1)!\,(n-k)!\,(n-k+1)!}$$

$$R^{gg}_{n+2}(\beta_1,\beta_2) = \sum_{k=0}^{n} (-1)^{k+1} \, \frac{n!\,(n+4)!\,\beta_1^k\,\beta_2^{n-k}}{k!\,(k+2)!\,(n-k)!\,(n-k+2)!}$$

$$R^{qg}_{n+\frac{3}{2}}(\beta_1,\beta_2) = \sum_{k=0}^{n} (-1)^{k} \, \frac{n!\,(n+3)!\,\beta_1^k\,\beta_2^{n-k}}{k!\,(k+1)!\,(n-k)!\,(n-k+2)!}\,, \tag{322}$$

where β_1 and β_2 are the Sudakov variables of two corresponding partons with spins s_1 and s_2, respectively. The degree n of the polynomials is related with the Lorenz spin j of the twist-2 operators by the relation

$$j = n + s_1 + s_2\,. \tag{323}$$

It is convenient, however, to pass to the momenta representation using the relation

$$N^{r_1\ldots r_l}(n_1,\ldots,n_l) = \int \prod_{t=1}^{l} \left(\frac{d\beta_t}{(n_t+1)!}\,\beta_t^{n_t} \right) \, N^{\{r\}}(\beta_1,\ldots,\beta_l)\,. \tag{324}$$

For these functions the evolution equation takes the form [30]

$$\frac{\partial}{\partial\tilde{\xi}}\, N^{r_1\ldots r_l}(n_1,\ldots n_l) =$$

$$\sum_{i<k} \sum_{r_{i'},r_{k'}} \sum_{n'_i,n'_k} \left[\Phi^{r_i r_k}_{r_{i'} r_{k'}}\right]^{n_i n_k}_{n_{i'} n_{k'}} \, \delta_{n_{i'}+n_{k'}+\Delta S_{ik},\, n_i+n_k} \, N^{r_1\ldots r_l}(n_1,\ldots,n_{i'}\ldots n_{k'}\ldots n_l)\,, \tag{325}$$

where

$$\Delta S_{ik} = s_{i'} + s_{k'} - s_i - s_k\,. \tag{326}$$

The new pair kernels are related to the kernels in the β-representation by the relations [30]

$$\sum_{n_{1'} n_{2'}} \left[\Phi^{r_1 r_2}_{r_{1'} r_{2'}}\right]^{n_1 n_2}_{n_{1'} n_{2'}} \, \frac{\beta_{1'}^{n_{1'}}}{(n_{1'}+1)!}\, \frac{\beta_{2'}^{n_{2'}}}{(n_{2'}+1)!}$$

$$= \int d\beta_1\, d\beta_2\, \frac{\beta_1^{n_1}}{(n_1+1)!}\, \frac{\beta_2^{n_2}}{(n_2+1)!}\, \delta(\beta_1+\beta_2-\beta_{1'}-\beta_{2'})\, \Phi^{r_i r_k}_{r_{i'} r_{k'}}(\beta_i,\beta_k|\beta_{i'},\beta_{k'})\,. \tag{327}$$

Due to the conformal invariance the kernels $\left[\Phi^{r_1 r_2}_{r_{1'} r_{2'}}\right]^{n_1 n_2}_{n_{1'} n_{2'}}$ are completely determined by their eigenfunctions and eigenvalues [30]

$$\left[\Phi^{r_1 r_2}_{r_{1'} r_{2'}}\right]^{n_1 n_2}_{n_{1'} n_{2'}} = \frac{a_{r_{1'}}(n_{1'})\, a_{r_{2'}}(n_{2'})}{a_{r_1}(n_1)\, a_{r_2}(n_2)} \sum_j \left(V^{r_1 r_2}\right)^j_{n_1 n_2} \left(V^{r_{1'} r_{2'}}\right)^j_{n_{1'} n_{2'}} \, \Lambda^{r_1 r_2}_{r_{1'} r_{2'}}(j) \, \left(\frac{j+2}{j-1}\right)^{\frac{1}{2}\Delta S_{12}}\,, \tag{328}$$

where ΔS_{12} was defined in eq.(326),

$$a_q(n) = \sqrt{n+1}\,,\;\; a_g(n) = \sqrt{\frac{n+1}{n+2}}\,, \tag{329}$$

and

$$(V^{r_1 r_2})^j_{n_1 n_2} = (-1)^{n_1+2s_1-1}\, C^{j,\Delta S}_{\frac{n_1+n_2}{2},\frac{n_2-n_1}{2};\,\frac{n_1+n_2}{2}+S,\frac{n_2-n_1}{2}+\Delta S}\,,\ \ S = s_1+s_2\,,\ \ \Delta S = s_1 - s_2\,. \tag{330}$$

The factor $C^{j,m}_{j_1,m_1;j_2,m_2}$ is the Clebsch-Gordon coefficient which does not depend on the color, flavor and helicity of the partons. Such dependence takes place only in the matrices $\Lambda^{r_1 r_2}_{r_{1'} r_{2'}}(j)$ acting on the corresponding indices of PCF. They are presented below (see [30])

$$\Lambda^{qq}_{qq}(j) = \frac{1}{2}\,(Q_6+Q_3)\left(\frac{N_c-1}{N_c}P_6 - \frac{N_c+1}{N_c}P_3\right)\left[(P_V+P_A+P_T)(2S_j-\frac{3}{2}) - \frac{P_V+P_A}{j(j+1)}\right]$$
$$+(-1)^j\,(Q_3\to -Q_3\,,\ P_3\to -P_3\,,\ P_A\to -P_A)\,, \tag{331}$$

$$\Lambda^{\bar q q}_{\bar q q}(j) = -(Q_1+Q_8)\left(\frac{N_c^2-1}{N_c}P_1 - \frac{1}{N_c}P_8\right)\left[(P_V+P_A+P_T)(2S_j-\frac{3}{2}) - \frac{P_V+P_A}{j(j+1)}\right]$$
$$-\frac{2}{3}\,n_f\,Q_1\,P_8\,P_V\,\delta_{j,1}\,, \tag{332}$$

$$\Lambda^{gg}_{gg}(j) = \left(N_c P_1 + P_{10} + P_{\overline{10}} - P_{27} + \frac{N_c}{2}P_{8d} + \frac{N_c}{2}P_{8f}\right)\left[(P_V+P_A+P_T)\left(\frac{11}{6} - 2S_j - \frac{n_f}{3N_c}\right)\right.$$
$$\left.+\frac{4}{j(j+1)}\left(P_V+P_A+\frac{3P_V}{(j-1)(j+2)}\right)\right] + (-1)^j\,(P_{8f}\to -P_{8f}\,,\ P_A\to -P_A)\,, \tag{333}$$

$$\Lambda^{gg}_{\bar q q}(j) = \sqrt{n_f}\,Q_1\left[\left(\sqrt{\frac{N_c^2-1}{N_c}}\,P_1 + \sqrt{\frac{N_c^2-4}{2N_c}}\,P_{8d} + i\sqrt{\frac{1}{2}N_c}\,P_{8f}\right)\frac{(j+2)(P_V+P_A)+\frac{4P_V}{j-1}}{j(j+1)}\right.$$
$$\left.+(-1)^j\,(P_{8f}\to -P_{8f}\,,\ P_A\to -P_A)\right]\,, \tag{334}$$

$$\Lambda^{\bar q q}_{gg}(j) = \frac{1}{2}\sqrt{n_f}\,Q_1\left[\left(\sqrt{\frac{N_c^2-1}{N_c}}\,P_1 + \sqrt{\frac{N_c^2-4}{2N_c}}\,P_{8d} - i\sqrt{\frac{1}{2}N_c}\,P_{8f}\right)\frac{(j-1)(P_V+P_A)+\frac{4P_V}{j+2}}{j(j+1)}\right.$$
$$\left.+(-1)^j\,(P_{8f}\to -P_{8f}\,,\ P_A\to -P_A)\right]\,, \tag{335}$$

$$\Lambda^{qg}_{qg}(j) = -\,(N_c P_3 + P_6 - P_{15})\left[(P_V+P_A+P_T)\left(2S_{j-\frac{1}{2}} + \frac{2}{2j+1} - \frac{5}{3} + \frac{n_f}{6N_c}\right)\right.$$
$$\left.-\frac{2(P_V+P_A)}{(j-\frac{1}{2})(j+\frac{3}{2})}\right] + \left(\frac{1}{N_c}P_3 + P_6 - P_{15}\right)\frac{(-1)^{j-\frac{1}{2}}}{j+\frac{1}{2}}\left[P_T - \frac{2(P_V+P_A)}{(j-\frac{1}{2})(j+\frac{3}{2})}\right]\,. \tag{336}$$

Here P_r $(r = V, A, T)$ are projectors for vector (V), axial (A) and tensor (T) helicity states of two partons

$$(P_V)^{\lambda_1\lambda_2}_{\lambda_1'\lambda_2'} = \frac{1}{2}\,\delta_{\lambda,-\lambda_2}\,\delta_{\lambda_1',-\lambda_2'}\,,\ \ (P_A)^{\lambda_1\lambda_2}_{\lambda_1'\lambda_2'} = \frac{1}{2}\,\lambda_1\,\lambda_1'\,\delta_{\lambda,-\lambda_2}\,\delta_{\lambda_1',-\lambda_2'}\,,$$
$$(P_T)^{\lambda_1\lambda_2}_{\lambda_1'\lambda_2'} = \frac{1}{2}\,(1+\lambda_1\,\lambda_1')\,\delta_{\lambda,\lambda_2}\,\delta_{\lambda_1',\lambda_2'}\,,\ \ \lambda_r = \pm 1\,. \tag{337}$$

The projectors P_r on the colour states $r = 1, 8f, 8d, 10, \overline{10}, 27$ (two gluons), $r = 3, 6$ (two quarks or anti-quarks), $r = 1, 8$ (quark-anti-quark pair) and $r = 3, 6, 15$ (gluon and quark or anti-quark) can be easily constructed (see for example ref. [33]). Similar expressions are

valid also for the projectors Q_r on the various flavour states $r = 1, 8$ (quark-anti-quark pair) and $r = 3, 6$ (two quarks or two anti-quarks) for the $SU(n_f)$-flavour group corresponding to massless u, d, s-quarks. We write below only projections for the transitions between two gluons and a quark-anti-quark pair

$$(P_1)^{ab}_{ij} = \Big(N_c(N_c^2-1)\Big)^{-\frac{1}{2}}\,\delta_{ab}\,\delta_{ij}\,,\ (P_{8d})^{ab}_{ij} = \left(\frac{2N_c}{N_c^2-4}\right)^{\frac{1}{2}} d_{abc}\,t^c_{ij}\,,$$

$$(P_{8f})^{ab}_{ij} = f_{abc}\,t^c_{ij}\,,\ (Q_1)^{f_1 f_2} = (n_f)^{-\frac{1}{2}}\,\delta_{f_1 f_2}\,,\ [t_a, t_b] = i f_{abc}\,t_c\,. \tag{338}$$

The above relations give a possibility to write the evolution equation for any QPO with the Lorentz spin j as a system of linear equations. The rank of this system grows rapidly with increasing j. The diagonalisation of the anomalous dimension matrix allows us to find the anomalous dimensions of the multiplicative renormalized operators. As an example, we write below this system for twist-3 operators responsible for the violation of the Bjorken scaling in the structure function $g_2(x)$.

7.10 Q^2-dependence of structure functions for the polarized target

The twist-3 operators appearing in the operator product expansion of two electro-magnetic currents j_μ and j_ν in the anti-symmetric tensor $T_{\mu\nu}$ (251) describing the electon scattering off the polarized target can be reduced to two quasi-partonic operators (302) with the use of equations of motion [29, 30]. To calculate their matrix elements (315) between hadron states one should introduce respectively two parton correlation functions

$$N(\beta_1, \beta_2, \beta_3) \equiv N^{\bar{q}qg}(\beta_1, \beta_2, \beta_3)\,,\ M(\beta_1, \beta_2, \beta_3) \equiv N^{ggg}(\beta_1, \beta_2, \beta_3)\,. \tag{339}$$

The evolution equations (319) are simplified in this case, as it will be shown below.

To begin with, we note, that the matrix elements of the operators (302) vanish if the signs of helicities of all three partons are the same. Due to the parity conservation it is enough to consider only the case, when two helicities are positive and one is negative. For the operators containing fermion fields the helicities of quark and anti-quark are opposite and therefore we can chose the positive sign for the gluon helicity. Moreover, PCF $N_8(\beta_1, \beta_2, \beta_3)$ with the flavour octet quantum number (for $SU(3)$-flavour group) do not mix with pure gluonic operators. Further, for the flavour singlet case the combination $N_1^{-++}(\beta_1, \beta_2, \beta_3) - N_1^{+-+}(\beta_2, \beta_1, \beta_3)$ has the negative charge parity and therefore it does not give any contribution to $T_{\mu\nu}$. Due to its positive charge parity the color structure of the gluonic operator is proportional to the antisymmetric structure constant f_{abc}. It means, that the Bose statistics gives us a possibility to express the corresponding correlation functions with various gluon helicities only in terms of $M^{-++}(\beta_1, \beta_2, \beta_3)$

$$M^{-++}(\beta_1, \beta_2, \beta_3) = -M^{-++}(\beta_1, \beta_3, \beta_3) = -M^{+-+}(\beta_2, \beta_1, \beta_3)$$

$$= -M^{++-}(\beta_3, \beta_2, \beta_1) = M^{++-}(\beta_2, \beta_3, \beta_1) = M^{+-+}(\beta_3, \beta_1, \beta_2)\,. \tag{340}$$

Thus, one should consider the following correlation functions

$$N_8(\beta_1, \beta_2, \beta_3) \equiv N_8^{-++}(\beta_1, \beta_2, \beta_3)\,,$$

$$N_1(\beta_1, \beta_2, \beta_3) \equiv N_1^{-++}(\beta_1, \beta_2, \beta_3) + N_1^{+-+}(\beta_2, \beta_1, \beta_3)\,,$$

$$M(\beta_1,\beta_2,\beta_3) \equiv M^{-++}(\beta_1,\beta_2,\beta_3)\,. \tag{341}$$

The evolution equations for N_8 in the momenta basis have the form (325) [29, 30]

$$\frac{\partial}{\partial\tilde{\xi}}\,N_8(n_1,n_2,n_3) = \left[\left(\Phi^{\bar{q}q}_{\bar{q}q}\right)_{V+A}\right]^{n_1n_2}_{n_{1'}n_{2'}} N_8(n_{1'},n_{2'},n_3) + \left[\left(\Phi^{\bar{q}q}_{\bar{q}q}\right)_{V-A}\right]^{n_1n_2}_{n_{1'}n_{2'}} N_8(n_{1'},n_{2'},n_3)$$

$$+\left[\left(\Phi^{\bar{q}g}_{\bar{q}g}\right)_{V+A}\right]^{n_1n_3}_{n_{1'}n_{3'}} N_8(n_{1'},n_2,n_{3'}) + \left[\left(\Phi^{qg}_{qg}\right)_{T}\right]^{n_2n_3}_{n_{2'}n_{3'}} N_8(n_1,n_{2'},n_{3'})\,, \tag{342}$$

where

$$\left(\Phi^{r_1r_2}_{r_{1'}r_{2'}}\right)_{V\pm A} \equiv \frac{1}{2}\left(\left(\Phi^{r_1r_2}_{r_{1'}r_{2'}}\right)_V \pm \left(\Phi^{r_1r_2}_{r_{1'}r_{2'}}\right)_A\right)\,. \tag{343}$$

The evolution equations for N_1 and M are more complicated [29, 30]

$$\frac{\partial}{\partial\tilde{\xi}}\,N_1(n_1,n_2,n_3) = \left[\left(\Phi^{\bar{q}q}_{\bar{q}q}\right)_{V+A}\right]^{n_1n_2}_{n_{1'}n_{2'}} N_1(n_{1'},n_{2'},n_3)$$

$$+\left[\left(\Phi^{\bar{q}q}_{\bar{q}q}\right)_{V-A}\right]^{n_1n_2}_{n_{1'}n_{2'}} N_1(n_{1'},n_{2'},n_3) + \left[\left(\Phi^{\bar{q}g}_{\bar{q}g}\right)_{V+A}\right]^{n_1n_3}_{n_{1'}n_{3'}} N_1(n_{1'},n_2,n_{3'})$$

$$+\left[\left(\Phi^{qg}_{qg}\right)_{T}\right]^{n_2n_3}_{n_{2'}n_{3'}} N_1(n_1,n_{2'},n_{3'}) + 4\left[\left(\Phi^{\bar{q}q}_{gg}\right)_{V+A}\right]^{n_1n_2}_{n_{1'}n_{2'}} M(n_{1'},n_{2'},n_3) \tag{344}$$

and

$$\frac{\partial}{\partial\tilde{\xi}}\,M(n_1,n_2,n_3) = 2\left[\left(\Phi^{gg}_{gg}\right)_{V+A}\right]^{n_1n_2}_{n_{1'}n_{2'}} M(n_{1'},n_{2'},n_3)$$

$$+2\left[\left(\Phi^{gg}_{gg}\right)_{V+A}\right]^{n_1n_3}_{n_{1'}n_{3'}} M(n_{1'},n_2,n_{3'}) + \left[\left(\Phi^{gg}_{gg}\right)_{T}\right]^{n_2n_3}_{n_{2'}n_{3'}} M(n_1,n_{2'},n_{3'})$$

$$+\left[\left(\Phi^{gg}_{\bar{q}q}\right)_{V+A}\right]^{n_1n_2}_{n_{1'}n_{2'}} N_1(n_{1'},n_{2'},n_3) + \left[\left(\Phi^{gg}_{\bar{q}q}\right)_{V+A}\right]^{n_1n_3}_{n_{1'}n_{3'}} N_1(n_{1'},n_2,n_{3'})\,, \tag{345}$$

where the sum over $n_{i'}, n_{k'}$ is implied with the additional constraint

$$n_{i'} + n_{k'} + \Delta S_{ik} = n_i + n_k\,. \tag{346}$$

Here S_{ik} is defined in eq. (326). The pair kernels $\left(\Phi^{r_1r_2}_{r_{1'}r_{2'}}\right)_s$ $(s = V, A, T)$ in the above equations are enlisted in eq. (328) and in the expressions for $\Lambda^{r_1r_2}_{r_{1'}r_{2'}}$ one should leave only the contributions containing the projectors (337) to the corresponding spin states V, A or T. In particular in the kernels describing the transition between partons $\bar{q}q$ and gg we have only the $V+A$ intermediate state in accordance with the fact, that the helicity of the third gluon is positive. The coefficients 4 and 2 in the equations for N_1 and M in front of some kernels appear due to our definition of N_1 and the symmetry relations for M (340). The color and flavor structures are not shown, because for each pair kernel only one color and flavor projector gives a non-zero contribution.

Let us consider for example the evolution equation for the singlet flavour state with j=3. In this case the independent components of momenta $N_1(n_1,n_2,n_3)$ and $M(n_1,n_2,n_3)$ are

$$x \equiv N_1(0,0,2)\,,\ y \equiv= N_1(0,1,1)\,,\ z \equiv N_1(1,0,1)\,,\ u \equiv N_1(2,0,0)\,,$$

and

$$v \equiv N_1(2,0,0)\,,\ w \equiv N_1(1,1,0)\,,\ m \equiv iM(0,1,0) = -iM(0,0,1)\,. \tag{347}$$

Using expressions (331)-(336), we can write the matrix of anomalous dimensions for the corresponding operators [30]. It turns out, that it contains an information about the anomalous dimension for lower spins $j = 1$ and $j = 2$. Indeed, because the total derivative from the local operator does not change its anomalous dimension, we obtain with the use of relation (324), that the linear combination

$$\sum_{k=1}^{l} (n_k + 2)\, N(n_1, ..., n_{k-1}, n_k + 1, n_{k+1}, ..., n_l) \tag{348}$$

has the same anomalous dimension matrix as PCF $N(n_1, ..., n_l)$ with lower j .

Therefore the components of $j = 2$ state

$$r = 2y + 3u + 2w\,,\ \ s = 2z + 3v + 2w\,,\ \ t = 3x + 2y + 2z \tag{349}$$

are closed under the Q^2-evolution [29, 30]

$$\frac{d}{d\bar{\xi}}\, r = \left(-\frac{3}{2}N_c + \frac{1}{N_c} - \frac{2}{3}n_f\right) r + \left(-\frac{2}{3N_c} - \frac{1}{3}n_f\right) s + \left(\frac{1}{3}N_c - \frac{1}{3N_c}\right) r\,,$$

$$\frac{d}{d\bar{\xi}}\, s = \left(-\frac{2}{3N_c} - \frac{1}{3}n_f\right) r + \left(-\frac{7}{4}N_c + \frac{1}{4N_c} - \frac{2}{3}n_f\right) s + \left(\frac{1}{2}N_c + \frac{1}{6N_c}\right) t\,,$$

$$\frac{d}{d\bar{\xi}}\, t = \left(-\frac{1}{2}N_c - \frac{1}{2N_c}\right) r + \left(\frac{3}{4}N_c + \frac{1}{4N_c}\right) s + \left(-\frac{11}{6}N_c - n_f\right) r\,. \tag{350}$$

In turn, one can separate from this system the evolution equation for the operator with $j = 1$

$$\frac{d}{d\bar{\xi}}\,(r + s + t) = \left(-N_c - \frac{1}{6N_c} - n_f\right)(r + s + t) \tag{351}$$

and therefore its large-q^2 behaviour is

$$r + s + t \sim \exp\left(\left(-N_c - \frac{1}{6N_c} - n_f\right)\bar{\xi}\right). \tag{352}$$

The components of the operators having really $j = 2$ are related as follows

$$t = -r - s\,, \tag{353}$$

where r and s satisfy the evolution equation

$$\frac{d}{d\bar{\xi}}\, r = \left(-\frac{11}{6}N_c - \frac{2}{3}n_f\right) r + \left(-\frac{1}{3}N_c - \frac{1}{3N_c} - \frac{1}{3}n_f\right) s\,,$$

$$\frac{d}{d\bar{\xi}}\, s = \left(-\frac{1}{2}N_c - \frac{5}{6N_c} - \frac{1}{3}n_f\right) r + \left(-\frac{9}{4}N_c + \frac{1}{12N_c} - \frac{2}{3}n_f\right) s\,. \tag{354}$$

Therefore the multiplicatively renormalized PCF with $j = 2$ depend on Q^2 as follows

$$\left(-\frac{1}{3}N_c - \frac{1}{3N_c} - \frac{1}{3}n_f\right) s + \left(-\frac{11}{6}N_c - \frac{2}{3}n_f - \lambda_1\right) r \sim e^{\lambda_2\bar{\xi}}\,,$$

$$\left(-\frac{1}{3}N_c - \frac{1}{3N_c} - \frac{1}{3}n_f\right) s + \left(-\frac{11}{6}N_c - \frac{2}{3}n_f - \lambda_2\right) \sim e^{\lambda_1\bar{\xi}}\,, \tag{355}$$

where $\lambda_{1,2}$ are two solutions of the secular equation

$$\left(-\frac{11}{6}N_c - \frac{2}{3}n_f - \lambda\right)\left(-\frac{9}{4}N_c + \frac{1}{12N_c} - \frac{2}{3}n_f - \lambda\right)$$
$$= \left(-\frac{1}{3}N_c - \frac{1}{3N_c} - \frac{1}{3}n_f\right)\left(-\frac{1}{2}N_c - \frac{5}{6N_c} - \frac{1}{3}n_f\right) . \tag{356}$$

In turn, the three components of PCF with $j = 3$ are expressed through independent ones

$$y = -\frac{3}{2}u - w\,,\ z = -\frac{3}{2}v - w\,,\ x = -\frac{2}{3}y - \frac{2}{3}z \tag{357}$$

which satisfy the evolution equations

$$\frac{d}{d\xi}u = \left(-\frac{29}{12}N_c + \frac{7}{6N_c} - \frac{8}{15}n_f\right)u$$
$$+\left(\frac{1}{12}N_c - \frac{1}{2N_c} - \frac{1}{5}n_f\right)v + \left(-\frac{1}{3}N_c - \frac{7}{6N_c} - \frac{4}{15}n_f\right) + \frac{17}{90}\sqrt{\frac{1}{2}N_c n_f}\,m\,,$$
$$\frac{d}{d\xi}v = \left(\frac{3}{20}N_c - \frac{3}{20N_c} - \frac{1}{5}n_f\right)u$$
$$+\left(-\frac{59}{20}N_c + \frac{37}{60N_c} - \frac{8}{15}n_f\right)v + \left(-\frac{2}{5}N_c - \frac{23}{30N_c} - \frac{4}{15}n_f\right) - \frac{1}{30}\sqrt{\frac{1}{2}N_c n_f}\,m\,,$$
$$\frac{d}{d\xi}w = \left(-\frac{3}{4}N_c - \frac{7}{8N_c} - \frac{1}{5}n_f\right)u$$
$$+\left(-\frac{1}{2}N_c - \frac{1}{8N_c} - \frac{1}{5}n_f\right)v + \left(-\frac{37}{12}N_c + \frac{3}{2N_c} - \frac{3}{5}n_f\right) - \frac{7}{60}\sqrt{\frac{1}{2}N_c n_f}\,m\,,$$
$$\frac{d}{d\xi}m = \sqrt{\frac{1}{2}N_c n_f}\left(\frac{37}{20}u - \frac{23}{20}v - \frac{7}{10}w\right) - \left(\frac{197}{60}N_c + n_f\right)m\,. \tag{358}$$

Again one can construct the multiplicatively renormalized PCF and find their anomalous dimensions by solving the corresponding secular equation.

For larger j the derivation of similar evolution equations is straightforward, but the construction of the multiplicatively normalized operators and their anomalous dimensions can be done only numerically, because the secular equation contains the polynomials $P_n(\lambda)$ of the rank $n > 4$.

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